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Stability and Synchronization of Discrete-Time Neural Networks With Switching Parameters and Time-Varying Delays

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Abstract—This paper is concerned with the problems of exponential stability analysis and synchronization of discrete-time switched delayed neural networks. Using the average dwell time approach together with the piecewise Lyapunov function technique, sufficient conditions are proposed to guarantee the exponential stability for the switched neural networks with time-delays. Benefitting from the delay partitioning method and the free-weighting matrix technique, the conservatism of the obtained results is reduced. In addition, the decay estimates are explicitly given and the synchronization problem is solved. The results reported in this paper not only depend upon the delay, but also depend upon the partitioning, which aims at reducing the conservatism. Numerical examples are presented to demonstrate the usefulness of the derived theoretical results.

Index Terms—Average dwell time, delay partitioning, delayed neural networks (DNNs), discrete time, exponential stability, switched parameters.

I. INTRODUCTION

DEDELAYED neural networks (DNNs) have received increasing attention in the past decades, because the neural networks have extensive applications in signal processing, image processing, speed detection of moving objects, and related areas. In addition, because of the finite speed of information processing, time-delay is frequently encountered, and the existence of a delay in a system may induce instability, oscillations or poor performances [17]. Therefore, many research results reported on DNNs in the literature; see [11], [18], [23], [37], [42], [49]–[52], and so on. Compared with continuous-time DNNs, the discrete-time DNNs have received relatively less attention. Some results can be found in the literature; see [9], [27], [34], [48], [53], [55], [56], and so on.

Recently, some research studied the DNNs with Markovian jumping parameters; see [21], [29], [35], [38], [43], and [44]. To mention a few, Huang *et al.* [21] investigated the robust

stability of stochastic delayed additive neural networks with Markovian switching; Liu *et al.* [29] studied stability and synchronization of discrete-time Markovian jumping neural networks with mixed mode-dependent time-delays; Rakkiyappan and Balasubramaniam [35] considered the dynamic analysis of Markovian jumping impulsive stochastic Cohen–Grossberg neural networks with discrete interval and distributed time-varying delays; Shen and Wang [38] investigated the almost sure exponential stability of recurrent neural networks with Markovian switching; and Wang *et al.* [43] addressed the exponential stability analysis of delayed recurrent neural networks with Markovian jumping parameters. Notice that all the above-mentioned results considered the Markovian jumping parameters. As it is well known that Markovian jumping can be seen as a special case of switching with specified probability distribution. Therefore, one common question could be: what properties do the DNNs have if the parameter switching not in the form of Markovian jumping? This question motivates us to carry out this research.

In this paper, we will conduct the research on the stability analysis and synchronization of discrete-time DNNs whose parameters are operated by a switching signal, that is, the discrete-time switched DNNs, which can be described by the following equation:

$$\begin{aligned} x(k+1) = & C(\alpha_k)x(k) + A(\alpha_k)f(x(k)) \\ & + A_d(\alpha_k)f(x(k-d(k))) \end{aligned} \quad (1)$$

where α_k is a switching signal that specifies which subsystem to be activated at a certain discrete-time instant. It should be noted, here, that the switching is arbitrary over the average dwell time, but not in the form of Markovian switching. To the best of our knowledge, there are a few results being reported on DNNs with arbitrary switching so far, see [2]–[5] [44]. The switched DNNs combining the theories of switched systems and neural networks are applied to high-speed signal processing, artificial intelligence, and gene selection in a DNA microarray analysis [10], [36], [41]. The continuous-time networks are usually discretized when they are used for the sake of computer-based simulation or experimentation. Unfortunately, the dynamic of the continuous-time networks cannot be preserved by discretization, as mentioned in [32]. In addition, the applications greatly depend on the stability of the equilibrium point of neural networks. Therefore, it is important to investigate the stability of the discrete-time switched DNNs. This paper will extend the results in [44] on continuous-time

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switched DNNs to the discrete-time case. The technique and methods to be used for the discrete-time case are, however, different from the continuous-time case. Investigating such switched DNNs would be rather difficult because the probability distribution of switching is not available. Many open questions still remain unsolved. Obviously, the key problem that deserves attention is to derive a condition, which can guarantee the stability of the switched DNN, when it changes from one mode to another under arbitrary switching signal with average dwell time. This presents an interesting and yet challenging research because it needs to integrate the research of the switched hybrid systems into that of the DNNs.

In the past few years, there has been an increasing interest in using the dwell time approach to handle switched systems [31]. Let $\tau_d > 0$ be the dwell time and $\mathcal{S}(\tau_d)$ be the set of all switching signals with interval between consecutive discontinuities being no smaller than τ_d . The result in [31] showed that a sufficiently large τ_d can be chosen to warrant the exponential stability of the underlying switched system for any switching signal belonging to $\mathcal{S}(\tau_d)$. Meanwhile, this concept was expanded in [19] to establish the average dwell time approach, which implies that the average time interval between consecutive switchings is no less than a specified constant τ_d^* . It was also shown in [19] that if such a constant τ_d^* is sufficiently large, then the switched system is exponentially stable. For discrete-time switched system, the results reported by using average dwell approach are very limited.

Time-delay, inducing instability and degrading system performance, is often encountered in practical engineering systems. For switched DNNs, the delay-dependent criteria for global robust periodicity analysis problem are obtained in terms of linear matrix inequalities (LMIs) by employing free-weighting matrix method in [30]. By introducing triple integral terms into a new Lyapunov functional, the robust passivity analysis of uncertain switched neural networks of neutral type with interval time-varying delay is considered in [33]. For the first time, the H_∞ weight learning law for switched Hopfield neural networks with time-delay was presented in [1]. Recently, the exponential stability analysis problem was investigated in [44] using delay partitioning method. The above-mentioned existing results are, however, concerned with continuous-time switched DNNs. There are a few works about discrete-time switched DNNs.

The delay partitioning technique (originally presented in [17]), also called as delay fractioning technique, is considered as an effective approach to reduce the conservatism of the stability condition of a time-delay linear system. The basic idea of this approach is to partition the time-delay (generally means time-invariant time-delay) into several components evenly, if the time-delay is time varying with lower bound, in this case, the lower bound is partitioned as several evenly spaced components. Construct a Lyapunov–Krasovskii functional (LKF) with consideration given to every delay component, it can be shown that the stability condition obtained by such a LKF is less conservative. Benefiting from this approach, many results for other problems are extended. To mention a few, the stability analysis for static recurrent neural networks with time-invariant delay was studied in [13],

the stability criteria for continuous systems with multiple time-varying delay components was proposed in [14], the stability analysis and stabilization of T-S fuzzy time-delay systems were investigated in [45] and [54], and the stabilization results for discrete singular delay systems were given in [15].

For synchronization problem of DNNs, there are some existing results [6]–[8], [24], [26], [57]. For example, some sufficient conditions for the exponential synchronization of DNNs were given in terms of LMIs using state feedback control in [16]. Using the drive-response concept, a delay-independent and decentralized control law for DNNs was derived to achieve the exponential synchronization in [12]. For DNNs with reaction-diffusion terms, some conditions dependent on the diffusion coefficients were given in [20] to guarantee the global synchronization under the impulsive controller. With the Lyapunov stability theory and the Kronecker product, a unified LMI approach was proposed to address the synchronization problem of discrete-time Markovian jumping neural networks with mixed time-delays in [29].

Unfortunately, the stability analysis and synchronization for discrete-time switched neural networks with time-varying delay were not fully investigated. How to deal with the discrete-time neural networks, switched system, and time-varying delay in a unified framework; how to reduce the conservatism of the result for stability condition of discrete-time neural networks with time-delay; and how to extend the result to the synchronization problem are remaining challenges.

The objective of this paper is to investigate the problems of the exponential stability and synchronization for discrete-time switched DNNs. Specifically, we will use the average dwell time approach and the piecewise Lyapunov function technique in this paper. We will consider two cases where the switched DNNs are with constant delay and time-varying delay, respectively. In both cases, we will introduce a new LKF to derive sufficient exponential stability conditions by integrating the delay partitioning method [17] with the free-weighting matrix technique. As such, the obtained stability conditions are delay dependent as well as partition dependent, which allow for conservatism reduction. With this, the corresponding synchronization method is proposed. The applicability of the derived analytical results is exemplified by several illustrative examples in comparison with the existing results.

Notations: The notations used throughout this paper are standard. \mathbb{R}^n is the n -dimensional Euclidean space, \mathbb{Z}^+ is the set of positive integers, the notation $P > 0$ means that P is real symmetric and positive definite, I and 0 are the identity matrix and a zero matrix, respectively, $\text{diag}\{\dots\}$ is a block-diagonal matrix, $\lambda_{\min}(P)$ ($\lambda_{\max}(P)$) is the minimum (maximum) eigenvalue of symmetric matrix P , and $\|\cdot\|$ is the Euclidean norm of a vector and its induced norm of a matrix. In symmetric block matrices or long matrix expressions, a star (\star) is used to represent a term that is induced by symmetry.

II. SYSTEM DESCRIPTION AND PRELIMINARIES

Consider a discrete-time n -neuron neural network with N modes described by the following delay-difference equation:

$$x(k+1) = Cx(k) + Ag(x(k)) + A_dg(x(k-d(k))) + J \quad (2)$$

for $k = 1, 2, \dots$, where $x(k) \triangleq [x_1(k), x_2(k), \dots, x_n(k)]^T \in \mathbb{R}^n$ is the neuron state vector, $J \triangleq [J_1, J_2, \dots, J_n]^T \in \mathbb{R}^n$ is a constant external input vector, $C \triangleq \text{diag}\{c_1, c_2, \dots, c_n\} > 0$ with $0 \leq c_i < 1$ is the state feedback coefficient matrix, $A, A_d \in \mathbb{R}^{n \times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively, and $g(x) \triangleq [g_1(x_1), g_2(x_2), \dots, g_n(x_n)]^T \in \mathbb{R}^n$ is the neuron activation function, which satisfies the following assumptions.

Assumption 1: The activation function $g(x)$ is a bounded function and satisfies $g(0) = 0$.

Assumption 2: For the activation function $g(x)$, there exist constants λ_i^- and λ_i^+ such that for $i \in 1, 2, \dots, n$

$$\lambda_i^- \leq \frac{g_i(x) - g_i(y)}{x - y} \leq \lambda_i^+ \quad \forall x, y \in \mathbb{R}, x \neq y. \quad (3)$$

The delay $d(k)$ satisfying either (A1) or (A2) below.

(A1) Constant time-delay: $1 \leq d(k) \equiv d$, $\forall k$, where d is a known integer.

(A2) Time-varying delay: $1 \leq d_1 \leq d(k) \leq d_2$, where d_1 and d_2 are two known integers representing the minimum and maximum delays, respectively.

Remark 1: Without loss of generality, it is assumed that $d_1 < d_2$ in (A2). When $d_1 = d_2$, it reduces to the constant delay case, which is the case in (A1).

Suppose $x^* = [x_1^*, x_2^*, \dots, x_n^*]^T$ is an equilibrium of (2). We shift the equilibrium to the origin by changing variables $\xi_i(k) = x_i(k) - x_i^*$ and $f_i(\xi_i(k)) = g_i(\xi_i(k) + x_i^*) - g_i(x_i^*)$, $i = 1, 2, \dots, n$. Then, the discrete-time DNN (2) is readily transformed into

$$\xi(k+1) = C\xi(k) + Af(\xi(k)) + A_d f(\xi(k-d(k))) \quad (4)$$

where

$$\begin{cases} \xi(k) = [\xi_1(k), \xi_2(k), \dots, \xi_n(k)]^T \\ f(\xi(k)) = [f_1(\xi_1(k)), f_2(\xi_2(k)), \dots, f_n(\xi_n(k))]^T. \end{cases}$$

As in [29], [47], and [56], the model of discrete-time switched DNN is described as follows:

$$\begin{aligned} \xi(k+1) &= C(\alpha_k)\xi(k) + A(\alpha_k)f(\xi(k)) \\ &\quad + A_d(\alpha_k)f(\xi(k-d(k))) \end{aligned} \quad (5)$$

where $\{(A(\alpha_k), A_d(\alpha_k), C(\alpha_k)) : \alpha_k \in \mathcal{N}\}$ is a family of matrices parameterized by an index set $\mathcal{N} = \{1, 2, \dots, N\}$ and $\alpha_k : \mathbb{Z}^+ \rightarrow \mathcal{N}$ is a piecewise constant function of time, called as a switching signal, which takes its values in the finite set \mathcal{N} . At an arbitrary discrete time k , the value of α_k , denoted by α for simplicity, might depend on k or $x(k)$, or both, or may be generated by any other hybrid scheme. We assume that the sequence of switching signal α_k is unpredictable, but its instantaneous value is available in real time. For the switching time sequence $k_0 < k_1 < k_2 < \dots$ of the switching signal α , the holding time between $[k_l, k_{l+1}]$ is called as the dwell time of the currently engaged subsystem, where $l \in \mathcal{N}$.

For each possible value $\alpha_k = i$, $i \in \mathcal{N}$, we will denote the system matrices associated with mode i by $A(i) = A(\alpha_k)$, $A_d(i) = A_d(\alpha_k)$, and $C(i) = C(\alpha_k)$, where $A(i)$, $A_d(i)$, and $C(i)$ are constant matrices. Corresponding to the switching signal α , we have the switching sequence

$\{(i_0, k_0), (i_1, k_1), \dots, (i_l, k_l), \dots, |i_l \in \mathcal{N}, l = 0, 1, \dots\}$ with $k_0 = 0$, which means that the i_l th subsystem is activated when $k \in [k_l, k_{l+1})$.

In addition, it is easily verified from (3) that $f_i(\xi_i(k))$ satisfies $f_i(0) = 0$ and $\forall \xi_i \neq 0$

$$\lambda_i^- \leq \frac{f_i(\xi_i(k))}{\xi_i(k)} \leq \lambda_i^+ \quad \forall i = 1, 2, \dots, n. \quad (6)$$

The following definitions and lemma are introduced, which will play the key roles in deriving our main results.

Definition 1: The equilibrium $\xi^* = 0$ of the discrete-time switched DNN in (5) is said to be exponentially stable under $\alpha(k)$ if the solution $\xi(k)$ satisfies the following:

$$\|\xi(k)\| \leq \eta \rho^{(k-k_0)} \|\xi(k_0)\|_{C^1} \quad \forall k \geq k_0$$

for constants $\eta \geq 1$ and $0 < \rho < 1$, and

$$\|\xi(k_0)\|_{C^1} \triangleq \max_{\theta=-\bar{d}, -\bar{d}+1, \dots, 0} \{\|\xi(k+\theta)\|, \|\zeta(k+\theta)\|\}$$

where $\zeta(\theta) \triangleq \xi(\theta+1) - \xi(\theta)$ and $\bar{d} \triangleq \max\{d_1, d_2\}$.

Lemma 1 [58]: For any constant matrix $M \in \mathbb{R}^{n \times n}$, $M > 0$, integers $a \leq b$, vector function $w : \{a, a+1, \dots, b\} \rightarrow \mathbb{R}^n$, then

$$-(b-a+1) \sum_{i=a}^b w^T(i) M w(i) \leq - \left(\sum_{i=a}^b w^T(i) \right) M \left(\sum_{i=a}^b w(i) \right).$$

Definition 2 ([28]): For any $T_2 > T_1 \geq 0$, let $N_a(T_1, T_2)$ denote the number of switchings of $\alpha(k)$ over (T_1, T_2) . If $N_a(T_1, T_2) \leq N_0 + (T_2 - T_1)/T_a$ holds for $T_a > 0$, $N_0 \geq 0$. Then, T_a is as called the average dwell time.

Remark 2: By average dwell time switching, we mean a class of switching signals such that the average time interval between consecutive switchings is at least T_a . Then, a basic problem for such systems is to specify the minimal T_a and obtain the admissible switching signals such that the underlying system is stable and satisfies a prescribed performance. As commonly used in the literature, we choose $N_0 = 0$ in Definition 2.

III. EXPONENTIAL STABILITY ANALYSIS

A. Constant Time-Delay Case

We shall consider the constant time-delay case, that is, (A1): $1 \leq d(k) \equiv d$, and we have the following result.

Theorem 1: Given integers $m \geq 1$, $\tau \geq 1$, and a constant $0 < \beta < 1$, supposed that there exist positive definite matrices $P(i) \in \mathbb{R}^{n \times n}$, $Q(i) \in \mathbb{R}^{m \times m}$, $R(i) \in \mathbb{R}^{n \times n}$, $\Pi = \text{diag}\{\pi_1, \pi_2, \dots, \pi_n\}$, $H = \text{diag}\{h_1, h_2, \dots, h_n\}$ and matrices $M(i)$, $N(i)$, $S(i)$ such that for $i \in \mathcal{N}$

$$\begin{aligned} \Psi(i) &\triangleq W_P^T \bar{P}(i) W_P + W_R^T \bar{R}(i) W_R + W_Q^T \bar{Q}(i) W_Q \\ &\quad + \text{sym} \left\{ W_{P1}^T P(i) W_{P2} + W_M^T(i) W_S(i) \right. \\ &\quad + W_F^T \Pi (\Lambda^+ + \Lambda^-) W_{P1} - W_{P1}^T \Pi \Lambda^+ \Lambda^- W_{P1} \\ &\quad - W_F^T \Pi W_F + W_{FD}^T H (\Lambda^+ + \Lambda^-) W_D \\ &\quad \left. - W_D^T H \Lambda^+ \Lambda^- W_D - W_{FD}^T H W_{FD} \right\} < 0 \end{aligned} \quad (7)$$

where

$$\begin{aligned}
\bar{P}(i) &\triangleq \begin{bmatrix} (1-\beta)P(i) & 0 \\ 0 & P(i) \end{bmatrix} \\
\bar{Q}(i) &\triangleq \begin{bmatrix} \beta Q(i) & 0 \\ 0 & -\beta^{\tau+1}Q(i) \end{bmatrix} \\
\bar{R}(i) &\triangleq \begin{bmatrix} \beta\tau R(i) & 0 \\ 0 & -\frac{\beta^{\tau+1}}{\tau}R(i) \end{bmatrix} \\
W_P &\triangleq \begin{bmatrix} I_n & 0_{n,(m+3)n} \\ 0_{n,(m+3)n} & I_n \end{bmatrix} \\
W_R &\triangleq \begin{bmatrix} 0_{n,(m+3)n} & I_n \\ I_n - I_n & 0_{n,(m+2)n} \end{bmatrix} \\
W_Q &\triangleq \begin{bmatrix} I_{mn} & 0_{mn,4n} \\ 0_{mn,n} & I_{mn} & 0_{mn,3n} \end{bmatrix} \\
W_{P1} &\triangleq [I_n \quad 0_{n,(m+3)n}] \\
W_{P2} &\triangleq [0_{n,(m+3)n} \quad I_n] \\
W_D &\triangleq [0_{n,mn} \quad I_n \quad 0_{n,3n}] \\
W_{FD} &\triangleq [0_{n,(m+2)n} \quad I_n \quad 0_{n,n}] \\
W_M(i) &\triangleq [M(i) \quad S(i) \quad 0_{n,(m+1)n} \quad N(i)] \\
W_S(i) &\triangleq [C(i) - I_n \quad 0_{n,mn} \quad A(i) \quad A_d(i) - I_n] \\
W_F &\triangleq [0_{n,(m+1)n} \quad I_n \quad 0_{n,2n}] \\
\Lambda^+ &\triangleq \text{diag}\{\lambda_1^+, \dots, \lambda_n^+\} \\
\Lambda^- &\triangleq \text{diag}\{\lambda_1^-, \dots, \lambda_n^-\}.
\end{aligned}$$

Then, the discrete-time switched DNN in (5) with time-delay satisfying (A1) is exponentially stable for any switching signal with average dwell time satisfying $T_a > T_a^* = \text{ceil}(-\ln \mu / \ln \beta)$, where function $\text{ceil}(a)$ represents rounding real number a to the nearest integer greater than or equal to a and $\mu \geq 1$ satisfies that $\forall i, j \in \mathcal{N}$

$$P(i) \leq \mu P(j), \quad Q(i) \leq \mu Q(j), \quad R(i) \leq \mu R(j). \quad (8)$$

Moreover, an estimate of the state decay is given by

$$\|\xi(k)\| \leq \eta \rho^{(k-k_0)} \|\xi(k_0)\|_{C^1} \quad (9)$$

where

$$\begin{aligned}
\rho &\triangleq \sqrt{\beta \mu^{1/T_a}}, \quad \eta \triangleq \sqrt{\frac{b}{a}} \geq 1, \quad a \triangleq \min_{\forall i \in \mathcal{N}} \lambda_{\min}(P(i)) \\
b &\triangleq \max_{\forall i \in \mathcal{N}} \lambda_{\max}(P(i)) + \tau \beta \max_{\forall i \in \mathcal{N}} \lambda_{\max}(Q(i)) \\
&\quad + \frac{\tau(\tau+1)}{2} \beta \max_{\forall i \in \mathcal{N}} \lambda_{\max}(R(i)). \quad (10)
\end{aligned}$$

Proof: By applying the delay-partitioning idea to the delay $d = m\tau$ that gives m parts, we construct the following Lyapunov–Krasovskii function:

$$V(\xi_k, \alpha_k) = \sum_{i=1}^3 V_i(\xi_k, \alpha_k) \quad (11)$$

with

$$\begin{cases} V_1(\xi_k, \alpha_k) \triangleq \xi^T(k) P(\alpha_k) \xi(k) \\ V_2(\xi_k, \alpha_k) \triangleq \sum_{l=k-\tau}^{k-1} \beta^{k-l} \zeta^T(l) Q(\alpha_k) \zeta(l) \\ V_3(\xi_k, \alpha_k) \triangleq \sum_{s=-\tau}^{-1} \sum_{l=k+s}^{k-1} \beta^{k-l} \varsigma^T(l) R(\alpha_k) \varsigma(l) \end{cases}$$

where $P(\alpha_k) > 0$, $Q(\alpha_k) > 0$, and $R(\alpha_k) > 0$ are real matrices to be determined, and

$$\zeta(l) \triangleq \begin{bmatrix} \xi(l) \\ \xi(l-\tau) \\ \xi(l-2\tau) \\ \vdots \\ \xi(l-m\tau+\tau) \end{bmatrix}, \quad \varsigma(l) \triangleq \xi(l+1) - \xi(l).$$

For $k \in [k_l, k_{l+1})$, we define $\Delta V_i(\xi_k, \alpha_k) \triangleq V_i(\xi_{k+1}, \alpha_k) - V_i(\xi_k, \alpha_k)$, $i = 1, 2, 3$ (Notice that the i_l th subsystem is activated when $k \in [k_l, k_{l+1})$, i.e., the system operates in one of the subsystems, thus increment of $V_i(\xi_k, \alpha_k)$ is defined for a fixed α_k). Then, $\Delta V(\xi_k, \alpha_k) = \sum_{i=1}^3 \Delta V_i(\xi_k, \alpha_k)$ with

$$\begin{aligned} \Delta V_1(\xi_k, \alpha_k) &= \xi^T(k+1) P(\alpha_k) \xi(k+1) - \xi^T(k) P(\alpha_k) \xi(k) \\ &= \varsigma^T(k) P(\alpha_k) \varsigma(k) + 2\zeta^T(k) P(\alpha_k) \varsigma(k) \end{aligned} \quad (12)$$

$$\begin{aligned} \Delta V_2(\xi_k, \alpha_k) &= -(1-\beta) \sum_{l=k-\tau}^{k-1} \beta^{k-l} \zeta^T(l) Q(\alpha_k) \zeta(l) \\ &\quad + \beta \zeta^T(k) Q(\alpha_k) \zeta(k) \\ &\quad - \beta^{\tau+1} \zeta^T(k-\tau) Q(\alpha_k) \zeta(k-\tau) \end{aligned} \quad (13)$$

$$\begin{aligned} \Delta V_3(\xi_k, \alpha_k) &= -(1-\beta) \sum_{s=-\tau}^{-1} \sum_{l=k+s}^{k-1} \beta^{k-l} \varsigma^T(l) R(\alpha_k) \varsigma(l) \\ &\quad + \beta \tau \varsigma^T(k) R(\alpha_k) \varsigma(k) \\ &\quad - \sum_{l=k-\tau}^{k-1} \beta^{k+1-l} \varsigma^T(l) R(\alpha_k) \varsigma(l). \end{aligned} \quad (14)$$

From Jensen's inequality in Lemma II, we can easily get

$$\begin{aligned} & - \sum_{l=k-\tau}^{k-1} \beta^{k+1-l} \varsigma^T(l) R(\alpha_k) \varsigma(l) \\ & \leq -\beta^{\tau+1} \sum_{l=k-\tau}^{k-1} \varsigma^T(l) R(\alpha_k) \varsigma(l) \\ & \leq -\frac{\beta^{\tau+1}}{\tau} [\zeta(k) - \zeta(k-\tau)]^T R(\alpha_k) [\zeta(k) - \zeta(k-\tau)]. \end{aligned} \quad (15)$$

Moreover, for any appropriately dimensioned matrices $M(\alpha_k)$, $N(\alpha_k)$, and $S(\alpha_k)$, $\alpha_k \in \mathcal{N}$, the following equation is true:

$$\begin{aligned} 0 &= 2 \left[\zeta^T(k) M^T(\alpha_k) + \varsigma^T(k) N^T(\alpha_k) \right. \\ &\quad \left. + \zeta^T(k-\tau) S^T(\alpha_k) \right] \\ &\quad \times \{ [C(\alpha_k) - I] \xi(k) - \varsigma(k) + A(\alpha_k) f(\xi(k)) \\ &\quad + A_d(\alpha_k) f(\xi(k-d)) \}. \end{aligned} \quad (16)$$

From (6), for any scalar $\pi_i > 0$ and $h_i > 0$, we have that for $i = 1, 2, \dots, n$

$$\begin{aligned} & 2 \sum_{i=1}^n \pi_i [f_i(\xi_i(k)) - \lambda_i^- \xi_i(k)] [f_i(\xi_i(k)) - \lambda_i^+ \xi_i(k)] \leq 0 \\ & 2 \sum_{i=1}^n h_i [f_i(\xi_i(k-d)) - \lambda_i^- \xi_i(k-d)] \\ & \quad \times [f_i(\xi_i(k-d)) - \lambda_i^+ \xi_i(k-d)] \leq 0 \end{aligned}$$

or equivalently

$$\left. \begin{aligned} 2f^T(\xi(k))\Pi(\Lambda^+ + \Lambda^-)\xi(k) - 2\xi^T(k)\Pi\Lambda^+\Lambda^-\xi(k) \\ - 2f^T(\xi(k))\Pi f(\xi(k)) \geq 0 \\ 2f^T(\xi(k-d))\mathbf{H}(\Lambda^+ + \Lambda^-)\xi(k-d) \\ - 2\xi^T(k-d)\mathbf{H}\Lambda^+\Lambda^-\xi(k-d) \\ - 2f^T(\xi(k-d))\mathbf{H}f(\xi(k-d)) \geq 0 \end{aligned} \right\} \quad (17)$$

where $\Lambda^+ \triangleq \text{diag}\{\lambda_1^+, \lambda_2^+, \dots, \lambda_n^+\}$ and $\Lambda^- \triangleq \text{diag}\{\lambda_1^-, \lambda_2^-, \dots, \lambda_n^-\}$.

Considering (12)–(17), it follows that:

$$\begin{aligned} \Delta V(\xi_k, \alpha_k) + (1 - \beta)V(\xi_k, \alpha_k) \\ \leq (1 - \beta)\xi^T(k)P(\alpha_k)\xi(k) \\ + 2\xi^T(k)P(\alpha_k)[\xi(k+1) - \xi(k)] \\ + [\xi(k+1) - \xi(k)]^T P(\alpha_k)[\xi(k+1) - \xi(k)] \\ + \beta\xi^T(k)Q(\alpha_k)\xi(k) - \beta^{\tau+1}\xi^T(k-\tau)Q(\alpha_k)\xi(k-\tau) \\ + \beta\tau\xi^T(k)R(\alpha_k)\xi(k) \\ - \frac{\beta^{\tau+1}}{\tau}[\xi(k) - \xi(k-\tau)]^T R(\alpha_k)[\xi(k) - \xi(k-\tau)] \\ + 2\left[\xi^T(k)M^T(\alpha_k) + \xi^T(k)N^T(\alpha_k) \right. \\ \left. + \xi^T(k-\tau)S^T(\alpha_k)\right] \\ \times \{[C(\alpha_k) - I]\xi(k) - \xi(k)\} \\ + A(\alpha_k)f(\xi(k)) + A_d(\alpha_k)f(\xi(k-d)) \\ + 2f^T(\xi(k))\Pi(\Lambda^+ + \Lambda^-)\xi(k) - 2\xi^T(k)\Pi\Lambda^+\Lambda^-\xi(k) \\ - 2f^T(\xi(k))\Pi f(\xi(k)) + 2f^T(\xi(k-d)) \\ \times \mathbf{H}(\Lambda^+ + \Lambda^-)\xi(k-d) \\ - 2\xi^T(k-d)\mathbf{H}\Lambda^+\Lambda^-\xi(k-d) - 2f^T(\xi(k-d)) \\ \times \mathbf{H}f(\xi(k-d)) \\ \triangleq \psi(k)^T \Psi(\alpha_k) \psi(k) \end{aligned}$$

where

$$\psi(k) \triangleq [\xi^T(k) \ \xi^T(k-d) \ f^T(\xi(k)) \ f^T(\xi(k-d)) \ \varsigma(k)]^T$$

and $\Psi(\alpha_k)$ is defined in (7).

On the other hand, (7) implies $\Psi(\alpha_k) < 0$. Then, we can easily achieve that for $k \in [k_l, k_{l+1})$,

$$\Delta V(\xi_k, \alpha_k) + (1 - \beta)V(\xi_k, \alpha_k) < 0. \quad (18)$$

Now, for an arbitrary piecewise constant switching signal α_k , and for any $k > 0$, we let $k_0 < k_1 < \dots < k_l < \dots$, $l = 1, \dots$, denote the switching points of α over the interval $(0, k)$. As mentioned earlier, the i th subsystem is activated when $k \in [k_l, k_{l+1})$. Therefore, for $k \in [k_l, k_{l+1})$, it holds from (18) that

$$\begin{aligned} V(\xi_{k+1}, \alpha_{k_l}) &< \beta V(\xi_k, \alpha_{k_l}) \\ &< \beta^2 V(\xi_{k-1}, \alpha_{k_l}) \\ &\leq \dots \\ &\leq \beta^{(k-k_l+1)} V(\xi_{k_l}, \alpha_{k_l}). \end{aligned}$$

Then, we have

$$V(\xi_k, \alpha_k) < \beta^{k-k_l} V(\xi_{k_l}, \alpha_{k_l}). \quad (19)$$

Using (8), at switching instant k_l , we have

$$\begin{aligned} P(\alpha_{k_l}) &\leq \mu P(\alpha_{k_{l-1}}), \quad Q(\alpha_{k_l}) \leq \mu Q(\alpha_{k_{l-1}}), \\ R(\alpha_{k_l}) &\leq \mu R(\alpha_{k_{l-1}}). \end{aligned}$$

Considering the Lyapunov functional in (11), at switching instant k_l , we have the following:

$$V(\xi_{k_l}, \alpha_{k_l}) \leq \mu V(\xi_{k_l}, \alpha_{k_{l-1}}). \quad (20)$$

Therefore, it follows from (19) and (20) and the relationship $\vartheta = N_\alpha(0, k) \leq (k - k_0)/T_a$ that

$$\begin{aligned} V(\xi_k, \alpha_k) &\leq \beta^{k-k_l} \mu V(\xi_{k_l}, \alpha_{k_{l-1}}) \\ &\leq \dots \\ &\leq \beta^{(k-k_0)} \mu^\vartheta V(\xi_{k_0}, \alpha_{k_0}) \\ &\leq (\beta \mu^{1/T_a})^{(k-k_0)} V(\xi_{k_0}, \alpha_{k_0}). \end{aligned} \quad (21)$$

Notice from (11) that

$$V(\xi_k, \alpha_k) \geq a \|\xi(k)\|^2, \quad V(\xi_{k_0}, \alpha_{k_0}) \leq b \|\xi(k_0)\|_{C^1}^2 \quad (22)$$

where a and b are defined in (10). Combining (21) and (22) yields

$$\|\xi(k)\|^2 \leq \frac{1}{a} V(\xi_k, \alpha_k) \leq \frac{b}{a} (\beta \mu^{1/T_a})^{(k-k_0)} \|\xi(k_0)\|_{C^1}^2. \quad (23)$$

Furthermore, letting $\rho \triangleq \sqrt{\beta \mu^{1/T_a}}$, it follows that:

$$\|\xi(k)\| \leq \sqrt{\frac{b}{a}} \rho^{(k-k_0)} \|\xi(k_0)\|_{C^1}. \quad (24)$$

By Definition 1, we know that if $0 < \rho < 1$, that is, $T_a > T_a^* = \text{ceil}(-\ln \mu / \ln \beta)$, the switched DNN in (5) is exponentially stable. The proof is completed. ■

Remark 3: Theorem 1 presents a sufficient condition for the exponential stability condition for the considered discrete-time switched DNN. Here, β plays a key role in controlling the low bound of the average dwell time, which can be seen from $T_a > T_a^* = \text{ceil}(-\ln \mu / \ln \beta)$. Specifically, if β is given a smaller value, the low bound of the average dwell time becomes smaller with a fixed μ , which may result in the instability of the system.

Remark 4: When $\mu = 1$ in $T_a > T_a^* = \text{ceil}(-\ln \mu / \ln \beta)$ we have $T_a > T_a^* = 0$, which means that the switching signal $\alpha(k)$ can be arbitrary. In this case, (8) turns out to be $P(i) = P(j) = P$, $Q(i) = Q(j) = Q$, $R(i) = R(j) = R$, $\forall i, j \in \mathcal{N}$, and the proposed approach becomes a quadratic one (a common Lyapunov functional for all subsystems).

Remark 5: The total number of decision variables $[(4 + 0.5m^2)n^2 + (1 + 0.5m)n]N + 2n$ in Theorem 1 is dependent on the delay partitioning number m , and it will increase if m increases. When the delay partitioning number $m \geq 1$ becomes larger, the conservatism of the results is further reduced. Therefore, an appropriate partitioning number m can be chosen to get the tradeoff between the number of decision variables and the less conservatism in practical.

Now, we give the following proposition to show that the proposed result will demonstrate its superiority in terms of the reduced conservatism with m increasing.

Proposition 1: Suppose that τ_m and d_m are the maximal τ and the maximal delays obtained by Theorem 1 for a given

number of partitions m , respectively. Then, for any positive integer r such that $(m/(m+r))\tau_m$ is an integer, we have $(m/(m+r))\tau_m \leq \tau_{m+r}$, and thus $d_m \leq d_{m+r}$.

Proof: From Theorem 1, we know that the following inequality holds for given partitioning number m and the integer τ_m :

$$\begin{aligned} & [\xi(k+1) - \xi(k)]^T P(\alpha_k) [\xi(k+1) - \xi(k)] \\ & - \frac{\beta^{\tau_m+1}}{\tau_m} [\xi(k) - \xi(k - \tau_m)]^T R(\alpha_k) [\xi(k) - \xi(k - \tau_m)] \\ & + 2f^T(\xi(k))\Pi(\Lambda^+ + \Lambda^-)\xi(k) - 2\xi^T(k)\Pi\Lambda^+\Lambda^-\xi(k) \\ & + (1 - \beta)\xi^T(k)P(\alpha_k)\xi(k) \\ & + 2\xi^T(k)P(\alpha_k) [\xi(k+1) - \xi(k)] \\ & - 2f^T(\xi(k))\Pi f(\xi(k)) \\ & + 2f^T(\xi(k-d))H(\Lambda^+ + \Lambda^-)\xi(k-d) \\ & - 2\xi^T(k-d)H\Lambda^+\Lambda^-\xi(k-d) \\ & - 2f^T(\xi(k-d))Hf(\xi(k-d)) \\ & + 2\left[\xi^T(k)M^T(\alpha_k) + \varsigma^T(k)N^T(\alpha_k) \right. \\ & \quad \left. + \xi^T(k - \tau_m)S^T(\alpha_k)\right] \\ & \times \{[C(\alpha_k) - I]\xi(k) - \varsigma(k) \\ & \quad + A(\alpha_k)f(\xi(k)) + A_d(\alpha_k)f(\xi(k-d))\} \\ & + \beta\xi^T(k)Q(\alpha_k)\xi(k) - \beta^{\tau_m+1}\xi^T(k - \tau_m) \\ & \quad \times Q(\alpha_k)\xi(k - \tau_m) + \beta\tau_m\varsigma^T(k)R(\alpha_k)\varsigma(k) < 0. \end{aligned}$$

Because $\Psi(i)$ is monotone increasing with respect to τ , the above inequality holds with τ_m replaced by $(m/(m+r))\tau_m$. The remaining parts of the proof can be carried out by following the similar lines as that of Proposition 1 in [15]. ■

When $\mu > 1$ and $\beta \rightarrow 0$, we have $T_a > T_a^* = 0$. On the other hand, when $\mu > 1$ and $\beta \rightarrow 1$, obviously we have $T_a \rightarrow \infty$, that is, there is no switching. In such a case, the discrete-time switched DNN in (5) is effectively operating at one of the subsystems all the time, and it turns out to be

$$\xi(k+1) = C\xi(k) + Af(\xi(k)) + A_d f(\xi(k-d)). \quad (25)$$

Corollary 1: Given integers $m \geq 1$ and $\tau \geq 1$, the discrete-time DNN in (25) is asymptotically stable if there exist positive definite matrices $P \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{mn \times mn}$, $R \in \mathbb{R}^{n \times n}$, $\Pi = \text{diag}\{\pi_1, \pi_2, \dots, \pi_n\}$, $H = \text{diag}\{h_1, h_2, \dots, h_n\}$, and matrices M , N , S such that

$$\begin{aligned} \Psi \triangleq & W_P^T \bar{P} W_P + W_R^T \bar{R} W_R + W_Q^T \bar{Q} W_Q + \text{sym} \left\{ W_{P1}^T P W_{P2} \right. \\ & + W_M^T W_S + W_F^T \Pi(\Lambda^+ + \Lambda^-) W_{P1} - W_{P1}^T \Pi \Lambda^+ \Lambda^- W_{P1} \\ & - W_F^T \Pi W_F + W_{FD}^T H(\Lambda^+ + \Lambda^-) W_D \\ & \left. - W_D^T H \Lambda^+ \Lambda^- W_D - W_{FD}^T H W_{FD} \right\} < 0 \end{aligned} \quad (26)$$

where

$$\begin{aligned} \bar{P} & \triangleq \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix}, \quad \bar{Q} \triangleq \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix}, \quad \bar{R} \triangleq \begin{bmatrix} \tau R & 0 \\ 0 & -\frac{1}{\tau} R \end{bmatrix} \\ W_M & \triangleq [M \ S \ 0_{n,(m+1)n} \ N] \\ W_S & \triangleq [C - I_n \ 0_{n,mn} \ A \ A_d - I_n] \end{aligned}$$

and W_P , W_R , W_Q , W_{P1} , W_{P2} , W_D , W_F , W_{FD} , Λ^+ , and Λ^- are defined in Theorem 1.

Proof: Choose the following Lyapunov–Krasovskii function:

$$\begin{aligned} \hat{V}(\xi_k) \triangleq & \xi^T(k)P\xi(k) + \sum_{l=k-\tau}^{k-1} \xi^T(l)Q\xi(l) \\ & + \sum_{s=-\tau}^{-1} \sum_{l=k+s}^{k-1} \varsigma^T(l)R\varsigma(l). \end{aligned}$$

The rest of the proof can be followed by the same lines of that of Theorem 1, thus we omit the details. ■

When $m = 1$ in deriving the result of Theorem 1, that is, we give up the delay partitioning approach. In such a case, we have the following result.

Corollary 2: Given a constant $0 < \beta < 1$ and an integer $d \geq 1$, supposed that there exist positive definite matrices $P(i) \in \mathbb{R}^{n \times n}$, $Q(i) \in \mathbb{R}^{n \times n}$, $R(i) \in \mathbb{R}^{n \times n}$, $\Pi = \text{diag}\{\pi_1, \pi_2, \dots, \pi_n\}$, $H = \text{diag}\{h_1, h_2, \dots, h_n\}$, and matrices $M(i)$, $N(i)$, $S(i)$ such that for $i \in \mathcal{N}$

$$\begin{bmatrix} \hat{\Psi}_{11}(i) & \hat{\Psi}_{12}(i) & \hat{\Psi}_{13}(i) & M^T(i)A_d(i) & \hat{\Psi}_{15}(i) \\ * & \hat{\Psi}_{22}(i) & S^T(i)A(i) & \hat{\Psi}_{24}(i) & -S^T(i) \\ * & * & -\Pi^T - \Pi & 0 & A^T(i)N(i) \\ * & * & * & -H^T - H & A_d^T(i)N(i) \\ * & * & * & * & \hat{\Psi}_{55}(i) \end{bmatrix} < 0$$

where

$$\begin{aligned} \hat{\Psi}_{11}(i) & \triangleq (1 - \beta)P(i) + \beta Q(i) - \frac{\beta^{d+1}}{d}R(i) \\ & \quad + \text{sym} \left\{ M^T(i)[C(i) - I] - \Pi\Lambda^+\Lambda^- \right\} \\ \hat{\Psi}_{12}(i) & \triangleq \frac{\beta^{d+1}}{d}R(i) + [C(i) - I]^T S(i) \\ \hat{\Psi}_{22}(i) & \triangleq -\beta^{d+1}Q(i) - \frac{\beta^{d+1}}{d}R(i) - 2H\Lambda^+\Lambda^- \\ \hat{\Psi}_{13}(i) & \triangleq M^T(i)A(i) + \Pi(\Lambda^+ + \Lambda^-) \\ \hat{\Psi}_{24}(i) & \triangleq S^T(i)A_d(i) + H(\Lambda^+ + \Lambda^-) \\ \hat{\Psi}_{15}(i) & \triangleq P(i) - M^T(i) + [C(i) - I]^T N(i) \\ \hat{\Psi}_{55}(i) & \triangleq P(i) + \beta d R(i) - N(i) - N^T(i). \end{aligned}$$

Then, the discrete-time switched DNN in (5) is exponentially stable for any switching signal with average dwell time satisfying $T_a > T_a^* = \text{ceil}(-\ln \mu / \ln \beta)$, where $\mu \geq 1$ satisfies (8).

B. Time-Varying Delay Case

In this section, we shall consider the time-varying delay case, that is, (A2): $0 < d_1 \leq d(k) \leq d_2$. We partition the time-delay $d(k)$ into two parts: constant part d_1 and time-varying part $\vartheta(k)$, that is, $d(k) = d_1 + \vartheta(k)$, where $\vartheta(k)$ satisfies $0 \leq \vartheta(k) \leq d_2 - d_1$. In this case, we set $d_1 = m\tau$, and have the following result.

Theorem 2: Given integers $m \geq 1$, $\tau \geq 1$, $m\tau < d_2$, and a constant $0 < \beta < 1$, supposed that there exist positive definite matrices $\mathcal{P}(i) \in \mathbb{R}^{n \times n}$, $\mathcal{Q}_1(i) \in \mathbb{R}^{mn \times mn}$, $\mathcal{Q}_2(i) \in \mathbb{R}^{n \times n}$, $\mathcal{Q}_3(i) \in \mathbb{R}^{n \times n}$, $\mathcal{R}_1(i) \in \mathbb{R}^{n \times n}$, $\mathcal{R}_2(i) \in \mathbb{R}^{n \times n}$, $\mathcal{Z}(i) \in \mathbb{R}^{n \times n}$,

$\Pi = \text{diag}\{\pi_1, \pi_2, \dots, \pi_n\}$, $V = \text{diag}\{v_1, v_2, \dots, v_n\}$, and matrices $\mathcal{M}(i)$, $\mathcal{N}(i)$, $\mathcal{S}(i)$ such that for $i \in \mathcal{N}$

$$\begin{aligned} \Phi_1(i) &\triangleq \mathcal{W}_P^T \bar{\mathcal{P}}(i) \mathcal{W}_P + \sum_{l=1}^2 \mathcal{W}_{Q_l}^T \bar{\mathcal{Q}}_l(i) \mathcal{W}_{Q_l} \\ &\quad + \sum_{l=1}^2 \mathcal{W}_{R_l}^T \bar{\mathcal{R}}_{1l}(i) \mathcal{W}_{R_l} + \mathcal{W}_Z^T \bar{\mathcal{Z}}(i) \mathcal{W}_Z \\ &\quad + \text{sym} \left\{ \mathcal{W}_{P_1}^T \mathcal{P}(i) \mathcal{W}_{P_2} + \mathcal{W}_M^T \mathcal{W}_S \right. \\ &\quad + \mathcal{W}_F^T \Pi (\Lambda^+ + \Lambda^-) \mathcal{W}_{P_1} - \mathcal{W}_{P_1}^T \Pi \Lambda^+ \Lambda^- \mathcal{W}_{P_1} \\ &\quad - \mathcal{W}_F^T \Pi \mathcal{W}_F + \mathcal{W}_{FD}^T V (\Lambda^+ + \Lambda^-) \mathcal{W}_{FD} \\ &\quad \left. - \mathcal{W}_D^T V \Lambda^+ \Lambda^- \mathcal{W}_D - \mathcal{W}_{FD}^T V \mathcal{W}_{FD} \right\} < 0 \end{aligned} \quad (27)$$

$$\begin{aligned} \Phi_2(i) &\triangleq \mathcal{W}_P^T \bar{\mathcal{P}}(i) \mathcal{W}_P + \sum_{l=1}^2 \mathcal{W}_{Q_l}^T \bar{\mathcal{Q}}_l(i) \mathcal{W}_{Q_l} \\ &\quad + \sum_{l=1}^2 \mathcal{W}_{R_l}^T \bar{\mathcal{R}}_{2l}(i) \mathcal{W}_{R_l} + \mathcal{W}_Z^T \bar{\mathcal{Z}}(i) \mathcal{W}_Z \\ &\quad + \text{sym} \left\{ \mathcal{W}_{P_1}^T \mathcal{P}(i) \mathcal{W}_{P_2} + \mathcal{W}_M^T \mathcal{W}_S \right. \\ &\quad + \mathcal{W}_F^T \Pi (\Lambda^+ + \Lambda^-) \mathcal{W}_{P_1} - \mathcal{W}_{P_1}^T \Pi \Lambda^+ \Lambda^- \mathcal{W}_{P_1} \\ &\quad - \mathcal{W}_F^T \Pi \mathcal{W}_F + \mathcal{W}_{FD}^T V (\Lambda^+ + \Lambda^-) \mathcal{W}_{FD} \\ &\quad \left. - \mathcal{W}_D^T V \Lambda^+ \Lambda^- \mathcal{W}_D - \mathcal{W}_{FD}^T V \mathcal{W}_{FD} \right\} < 0 \end{aligned} \quad (28)$$

where $\tilde{\beta} \triangleq (\beta^{d_2+1})/(d_2 - m\tau)$ and

$$\begin{aligned} \bar{\mathcal{P}}(i) &\triangleq \text{diag}\{(1 - \beta) \mathcal{P}(i), \mathcal{P}(i)\} \\ \bar{\mathcal{Q}}_1(i) &\triangleq \text{diag}\left\{\beta \mathcal{Q}_1(i), -\beta^{\tau+1} \mathcal{Q}_1(i)\right\} \\ \bar{\mathcal{Q}}_2(i) &\triangleq \text{diag}\left\{\beta \mathcal{Q}_2(i), -\beta^{d_2+1} \mathcal{Q}_2(i)\right\} \\ \bar{\mathcal{R}}_{11}(i) &\triangleq \text{diag}\left\{\beta \tau \mathcal{R}_1(i), -\frac{\beta^{\tau+1}}{\tau} \mathcal{R}_1(i)\right\} \\ \bar{\mathcal{R}}_{21}(i) &\triangleq \text{diag}\left\{\beta \tau \mathcal{R}_1(i), -\frac{\beta^{\tau+1}}{\tau} \mathcal{R}_1(i)\right\} \\ \bar{\mathcal{R}}_{12}(i) &\triangleq \text{diag}\left\{\beta(d_2 - m\tau) \mathcal{R}_2(i), -2\tilde{\beta} \mathcal{R}_2(i), -\tilde{\beta} \mathcal{R}_2(i)\right\} \\ \bar{\mathcal{R}}_{22}(i) &\triangleq \text{diag}\left\{\beta(d_2 - m\tau) \mathcal{R}_2(i), -\tilde{\beta} \mathcal{R}_2(i), -2\tilde{\beta} \mathcal{R}_2(i)\right\} \\ \bar{\mathcal{Z}}(i) &\triangleq \text{diag}\left\{\beta(d_2 - m\tau + 1) \mathcal{Z}(i), -\beta^{d_2+1} \mathcal{Z}(i)\right\} \\ \mathcal{W}_P &\triangleq \begin{bmatrix} I_n & 0_{n,(m+5)n} \\ 0_{n,(m+5)n} & I_n \end{bmatrix} \\ \mathcal{W}_{Q_1} &\triangleq \begin{bmatrix} I_{mn} & 0_{mn,6n} \\ 0_{mn,n} & I_{mn} & 0_{mn,5n} \end{bmatrix} \\ \mathcal{W}_{Q_2} &\triangleq \begin{bmatrix} 0_{n,(m+5)n} & I_n \\ 0_{n,(m+2)n} & I_n & 0_{n,3n} \end{bmatrix} \\ \mathcal{W}_Z &\triangleq \begin{bmatrix} I_n & 0_{n,(m+5)n} \\ 0_{n,(m+1)n} & I_n & 0_{n,4n} \end{bmatrix} \\ \mathcal{W}_{R_1} &\triangleq \begin{bmatrix} 0_{n,(m+5)n} & I_n \\ I_n & -I_n & 0_{n,(m+4)n} \end{bmatrix} \\ \mathcal{W}_{R_2} &\triangleq \begin{bmatrix} 0_{n,(m+5)n} & I_n \\ 0_{n,(m+1)n} & I_n & -I_n & 0_{n,3n} \\ 0_{n,mn} & I_n & -I_n & 0_{n,4n} \end{bmatrix} \\ \mathcal{W}_M(i) &\triangleq [\mathcal{M}(i) \ \mathcal{S}(i) \ 0_{n,(m+3)n} \ \mathcal{N}(i)] \\ \mathcal{W}_{P_1} &\triangleq [I_n \ 0_{n,(m+5)n}], \ \mathcal{W}_{P_2} \triangleq [0_{n,(m+5)n} \ I_n] \\ \mathcal{W}_S(i) &\triangleq [C(i) - I_n \ 0_{n,(m+2)n} \ A(i) \ A_d(i) - I_n] \end{aligned}$$

$$\begin{aligned} \mathcal{W}_F &\triangleq [0_{n,(m+3)} \ I_n \ 0_{n,2n}] \\ \mathcal{W}_D &\triangleq [0_{n,(m+1)n} \ I_n \ 0_{n,4n}] \\ \mathcal{W}_{FD} &\triangleq [0_{n,(m+4)n} \ I_n \ 0_{n,n}]. \end{aligned}$$

Then, the discrete-time switched DNN in (5) with time-delay satisfying (A2) is exponentially stable for any switching signal with average dwell time satisfying $T_a > T_a^* = \text{ceil}(-\ln \mu / \ln \beta)$, where $\mu \geq 1$ satisfies that $\forall i, j \in \mathcal{N}$

$$\begin{aligned} \mathcal{P}(i) &\leq \mu \mathcal{P}(j), \quad \mathcal{Q}_\ell(i) \leq \mu \mathcal{Q}_\ell(j), \quad \ell = 1, 2, 3 \\ \mathcal{R}_v(i) &\leq \mu \mathcal{R}_v(j), \quad v = 1, 2. \end{aligned} \quad (29)$$

Proof: Choose a Lyapunov function of the form as follows:

$$W(\zeta_k, \alpha_k) \triangleq \sum_{j=1}^4 W_j(\zeta_k, \alpha_k) \quad (30)$$

with

$$\begin{cases} W_1(\zeta_k, \alpha_k) \triangleq \zeta^T(k) \mathcal{P}(\alpha_k) \zeta(k) \\ W_2(\zeta_k, \alpha_k) \triangleq \sum_{l=k-\tau}^{k-1} \beta^{k-l} \varphi^T(l) \mathcal{Q}_1(\alpha_k) \varphi(l) \\ \quad + \sum_{l=k-d_2}^{k-1} \beta^{k-l} \zeta^T(l) \mathcal{Q}_2(\alpha_k) \zeta(l) \\ W_3(\zeta_k, \alpha_k) \triangleq \sum_{s=-\tau}^{-1} \sum_{l=k+s}^{k-1} \beta^{k-l} \varsigma^T(l) \mathcal{R}_1(\alpha_k) \varsigma(l) \\ \quad + \sum_{s=-d_2}^{-m\tau-1} \sum_{l=k+s}^{k-1} \beta^{k-l} \varsigma^T(l) \mathcal{R}_2(\alpha_k) \varsigma(l) \\ W_4(\zeta_k, \alpha_k) \triangleq \sum_{s=-d_2+1}^{-m\tau+1} \sum_{l=k+1+s}^{k-1} \beta^{k-l} \zeta^T(l) \mathcal{Z}(\alpha_k) \zeta(l) \end{cases}$$

where

$$\varphi(l) \triangleq \begin{bmatrix} \zeta(l) \\ \zeta(l - \tau) \\ \zeta(l - 2\tau) \\ \vdots \\ \zeta(l - (m-1)\tau) \end{bmatrix}, \quad \varsigma(l) \triangleq \zeta(l+1) - \zeta(l)$$

and $\mathcal{P}(\alpha_k) > 0$, $\mathcal{S}(\alpha_k) > 0$, $\mathcal{Q}_1(\alpha_k) > 0$, $\mathcal{Q}_2(\alpha_k) > 0$, $\mathcal{R}_1(\alpha_k) > 0$, and $\mathcal{R}_2(\alpha_k) > 0$ are real matrices to be determined.

For $k \in [k_l, k_{l+1})$, as in the previous section, we define $\Delta W_j(\zeta_k, \alpha_k) \triangleq W_j(\zeta_{k+1}, \alpha_k) - W_j(\zeta_k, \alpha_k)$, $j = 1, 2, 3, 4$, thus we have $\Delta W(\zeta_k, \alpha_k) = \sum_{j=1}^4 \Delta W_j(\zeta_k, \alpha_k)$ with

$$\begin{aligned} \Delta W(\zeta_k, \alpha_k) &= \sum_{j=1}^4 \Delta W_j(\zeta_k, \alpha_k) \\ \Delta W_1(\zeta_k, \alpha_k) &= \zeta^T(k+1) \mathcal{P}(\alpha_k) \zeta(k+1) - \zeta^T(k) \mathcal{P}(\alpha_k) \zeta(k) \\ &= \varsigma^T(k) \mathcal{P}(\alpha_k) \varsigma(k) + 2\zeta^T(k) \mathcal{P}(\alpha_k) \varsigma(k) \quad (31) \\ \Delta W_2(\zeta_k, \alpha_k) &= -(1 - \beta) \sum_{l=k-\tau}^{k-1} \beta^{k-l} \varphi^T(l) \mathcal{Q}_1(\alpha_k) \varphi(l) \\ &\quad - (1 - \beta) \sum_{l=k-d_2}^{k-1} \beta^{k-l} \zeta^T(l) \mathcal{Q}_2(\alpha_k) \zeta(l) \\ &\quad + \beta \varphi^T(k) \mathcal{Q}_1(\alpha_k) \varphi(k) \\ &\quad - \beta^{\tau+1} \varphi^T(k - \tau) \mathcal{Q}_1(\alpha_k) \varphi(k - \tau) \end{aligned}$$

$$\begin{aligned}
& +\beta \xi^T(k) \mathcal{Q}_2(a_k) \xi(k) \\
& -\beta^{d_2+1} \xi^T(k-d_2) \mathcal{Q}_2(a_k) \xi(k-d_2) \quad (32) \\
\Delta W_3(\xi_k, a_k) = & -(1-\beta) \sum_{s=-\tau}^{-1} \sum_{l=k+s}^{k-1} \beta^{k-l} \xi^T(l) \mathcal{R}_1(a_k) \xi(l) \\
& -(1-\beta) \sum_{s=-d_2}^{-m\tau-1} \sum_{l=k+s}^{k-1} \beta^{k-l} \xi^T(l) \mathcal{R}_2(a_k) \xi(l) \\
& +\beta \tau \xi^T(k) \mathcal{R}_1(a_k) \xi(k) \\
& -\sum_{l=k-\tau}^{k-1} \beta^{k+1-l} \xi^T(l) \mathcal{R}_1(a_k) \xi(l) \\
& +\beta(d_2-m\tau) \xi^T(k) \mathcal{R}_2(a_k) \xi(k) \\
& -\sum_{l=k-d_2}^{k-m\tau-1} \beta^{k+1-l} \xi^T(l) \mathcal{R}_2(a_k) \xi(l) \quad (33) \\
\Delta W_4(\xi_k, a_k) \leq & -(1-\beta) \sum_{s=-d_2+1}^{-m\tau+1} \sum_{l=k-1+s}^{k-1} \beta^{k-l} \xi^T(l) \\
& \times \mathcal{L}(a_k) \xi(l) \\
& +\beta(d_2-m\tau+1) \xi^T(k) \mathcal{L}(a_k) \xi(k) \\
& -\beta^{d_2+1} \xi^T(k-d(k)) \mathcal{L}(a_k) \xi(k-d(k)). \quad (34)
\end{aligned}$$

Letting $\gamma = (d(k) - m\tau)/(d_2 - m\tau)$ and by Jensen's inequality, we have the following:

$$\begin{aligned}
& -\sum_{l=k-\tau}^{k-1} \beta^{k+1-l} \xi^T(l) \mathcal{R}_1(a_k) \xi(l) \\
& \leq -\beta^{1+\tau} \sum_{l=k-\tau}^{k-1} \xi^T(l) \mathcal{R}_1(a_k) \xi(l) \\
& \leq -\frac{\beta^{1+\tau}}{\tau} [\xi(k) - \xi(k-\tau)]^T \mathcal{R}_1(a_k) [\xi(k) - \xi(k-\tau)] \quad (35)
\end{aligned}$$

and

$$\begin{aligned}
& -\sum_{l=k-d_2}^{k-m\tau-1} \beta^{k+1-l} \xi^T(l) \mathcal{R}_2(a_k) \xi(l) \\
& = -\sum_{l=k-d_2}^{k-d(k)-1} \beta^{k+1-l} \xi^T(l) \mathcal{R}_2(a_k) \xi(l) \\
& \quad -\sum_{l=k-d(k)}^{k-m\tau-1} \beta^{k+1-l} \xi^T(l) \mathcal{R}_2(a_k) \xi(l) \\
& \leq -\sum_{l=k-d_2}^{k-d(k)-1} \beta^{k+1-l} (d_2-d(k)) \xi^T(l) \frac{\mathcal{R}_2(a_k)}{d_2-m\tau} \xi(l) \\
& \quad -\sum_{l=k-d_2}^{k-d(k)-1} \gamma \beta^{k+1-l} (d_2-d(k)) \xi^T(l) \frac{\mathcal{R}_2(a_k)}{d_2-m\tau} \xi(l) \\
& \quad -\sum_{l=k-d(k)}^{k-m\tau-1} \beta^{k+1-l} (d(k)-m\tau) \xi^T(l) \frac{\mathcal{R}_2(a_k)}{d_2-m\tau} \xi(l)
\end{aligned}$$

$$\begin{aligned}
& -\sum_{l=k-d(k)}^{k-m\tau-1} (1-\gamma) \beta^{k+1-l} (d(k)-m\tau) \xi^T(l) \\
& \quad \times \frac{\mathcal{R}_2(a_k)}{d_2-m\tau} \xi(l) \\
& \leq -(1+\gamma) \tilde{\beta} \xi_1^T(k) \mathcal{R}_2(a_k) \xi_1(k) \\
& \quad - (2-\gamma) \tilde{\beta} \xi_2^T(k) \mathcal{R}_2(a_k) \xi_2(k) \quad (36)
\end{aligned}$$

where $\xi_1(k) \triangleq \xi(k-d(k)) - \xi(k-d_2)$ and $\xi_2(k) \triangleq \xi(k-m\tau) - \xi(k-d(k))$.

Moreover, for any appropriately dimensioned matrices $\mathcal{M}(a_k)$, $\mathcal{N}(a_k)$, and $\mathcal{S}(a_k)$, $a_k \in \mathcal{N}$, we have the following:

$$\begin{aligned}
0 = & 2 \left[\xi^T(k) \mathcal{M}(a_k) + \xi^T(k) \mathcal{N}(a_k) + \xi^T(k-\tau) \mathcal{S}(a_k) \right] \\
& \times \{ [C(a_k) - I] \xi(k) - \xi(k) \\
& + A(a_k) f(\xi(k)) + A_d(a_k) f(\xi(k-d(k))) \}. \quad (37)
\end{aligned}$$

From (6), for any $\Pi = \text{diag}\{\pi_1, \pi_2, \dots, \pi_n\} > 0$ and $V = \text{diag}\{v_1, v_2, \dots, v_n\} > 0$, we have the following:

$$\left. \begin{aligned}
& 2f^T(\xi(k)) \Pi (\Lambda^+ + \Lambda^-) \xi(k) - 2\xi^T(k) \Pi \Lambda^+ \Lambda^- \xi(k) \\
& \quad - 2f^T(\xi(k)) \Pi f(\xi(k)) \geq 0 \\
& 2f^T(\xi(k-d(k))) V (\Lambda^+ + \Lambda^-) \xi(k-d(k)) \\
& \quad - 2\xi^T(k-d(k)) V \Lambda^+ \Lambda^- \xi(k-d(k)) \\
& \quad - 2f^T(\xi(k-d(k))) V f(\xi(k-d(k))) \geq 0.
\end{aligned} \right\} \quad (38)$$

Considering (31)–(36) and (38), we have

$$\begin{aligned}
& \Delta W(\xi_k, a_k) + (1-\beta) W(\xi_k, a_k) \\
& \leq \phi^T(k) [\gamma \Phi_1(a_k) + (1-\gamma) \Phi_2(a_k)] \phi(k)
\end{aligned}$$

where $\phi(k) \triangleq [\varphi^T(k) \quad \xi^T(k-d_1) \quad \xi^T(k-d(k)) \quad \xi^T(k-d_2) \quad f^T(\xi(k)) \quad f^T(\xi(k-d(k))) \quad \xi^T(k)]^T$, and $\Phi_1(a_k)$, $\Phi_2(a_k)$ are defined in (27) and (28), respectively. Moreover, from $\Phi_1(a_k) < 0$ and $\Phi_2(a_k) < 0$, it follows that $\forall k \in [k_l, k_{l+1})$:

$$\begin{aligned}
& \Delta W(\xi_k, a_k) + (1-\beta) W(\xi_k, a_k) \\
& \leq \phi^T(k) [\gamma \Phi_1(a_k) + (1-\gamma) \Phi_2(a_k)] \phi(k) < 0.
\end{aligned}$$

The rest of the proof can be followed by the same lines of the proof of Theorem 1. The proof is completed. ■

Remark 6: The technique of dealing with the term $-\sum_{l=k-d_2}^{k-m\tau-1} \beta^{k+1-l} \xi^T(l) \mathcal{R}_2(a_k) \xi(l)$ came from [22], which has reduced the conservatism of the results in previous work.

Remark 7: Because of the instrumental idea of delay partitioning, the reduction of conservatism becomes more obvious with the partitioning getting thinner (that is, m becoming bigger). Simultaneously, the number of decision variables in the obtained conditions, however, will be quickly increased as m increases. It is difficult to determine the maximum value of the delay bound of the system in (4) because of the time-varying and nonlinear nature of the switched system.

In addition, we have $T_a \rightarrow \infty$ (which means that there is no switching) when $\mu > 1$ and $\beta \rightarrow 1$. In such a case, the discrete-time switched DNN in (5) is effectively operating at one of the subsystems all the time, and it turns out to be the form as follows:

$$\xi(k+1) = C\xi(k) + Af(\xi(k)) + A_d f(\xi(k-d(k))). \quad (39)$$

In the following, we will give a corollary on the stability of the discrete-time DNN in (39).

Corollary 3: Given an integer $m \geq 1$, $\tau \geq 1$, and $m\tau < d_2$, the discrete-time DNN in (39) with time-delay satisfying (A2) is asymptotically stable, if there exist positive definite matrices $\mathcal{P} \in \mathbb{R}^{n \times n}$, $\mathcal{Q}_1 \in \mathbb{R}^{mn \times mn}$, $\mathcal{Q}_2 \in \mathbb{R}^{n \times n}$, $\mathcal{Q}_3 \in \mathbb{R}^{n \times n}$, $\mathcal{R}_1 \in \mathbb{R}^{n \times n}$, $\mathcal{R}_2 \in \mathbb{R}^{n \times n}$, $\mathcal{Z} \in \mathbb{R}^{n \times n}$, $\Pi = \text{diag}\{\pi_1, \pi_2, \dots, \pi_n\}$, $V = \text{diag}\{v_1, v_2, \dots, v_n\}$, and matrices \mathcal{M} , \mathcal{N} , and \mathcal{S} such that

$$\begin{aligned} \hat{\Phi}_1 \triangleq & \mathcal{W}_P^T \bar{\mathcal{P}} \mathcal{W}_P + \sum_{i=1}^2 \mathcal{W}_{Q_i}^T \bar{\mathcal{Q}}_i \mathcal{W}_{Q_i} + \sum_{i=1}^2 \mathcal{W}_{R_i}^T \bar{\mathcal{R}}_{1i} \mathcal{W}_{R_i} \\ & + \mathcal{W}_Z^T \bar{\mathcal{Z}} \mathcal{W}_Z + \text{sym} \left\{ \mathcal{W}_{P_1}^T \mathcal{P} \mathcal{W}_{P_2} + \mathcal{W}_M^T \mathcal{W}_S \right. \\ & + \mathcal{W}_F^T \Pi (\Lambda^+ + \Lambda^-) \mathcal{W}_{P_1} - \mathcal{W}_{P_1}^T \Pi \Lambda^+ \Lambda^- \mathcal{W}_{P_1} \\ & - \mathcal{W}_F^T \Pi \mathcal{W}_F + \mathcal{W}_{FD}^T V (\Lambda^+ + \Lambda^-) \mathcal{W}_{FD} \\ & \left. - \mathcal{W}_D^T V \Lambda^+ \Lambda^- \mathcal{W}_D - \mathcal{W}_{FD}^T V \mathcal{W}_{FD} \right\} < 0 \end{aligned} \quad (40)$$

$$\begin{aligned} \hat{\Phi}_2 \triangleq & \mathcal{W}_P^T \bar{\mathcal{P}} \mathcal{W}_P + \sum_{i=1}^2 \mathcal{W}_{Q_i}^T \bar{\mathcal{Q}}_i \mathcal{W}_{Q_i} + \sum_{i=1}^2 \mathcal{W}_{R_i}^T \bar{\mathcal{R}}_{2i} \mathcal{W}_{R_i} \\ & + \mathcal{W}_Z^T \bar{\mathcal{Z}} \mathcal{W}_Z + \text{sym} \left\{ \mathcal{W}_{P_1}^T \mathcal{P} \mathcal{W}_{P_2} + \mathcal{W}_M^T \mathcal{W}_S \right. \\ & + \mathcal{W}_F^T \Pi (\Lambda^+ + \Lambda^-) \mathcal{W}_{P_1} - \mathcal{W}_{P_1}^T \Pi \Lambda^+ \Lambda^- \mathcal{W}_{P_1} \\ & - \mathcal{W}_F^T \Pi \mathcal{W}_F + \mathcal{W}_{FD}^T V (\Lambda^+ + \Lambda^-) \mathcal{W}_{FD} \\ & \left. - \mathcal{W}_D^T V \Lambda^+ \Lambda^- \mathcal{W}_D - \mathcal{W}_{FD}^T V \mathcal{W}_{FD} \right\} < 0 \end{aligned} \quad (41)$$

where $\mathcal{W}_P, \mathcal{W}_{P_1}, \mathcal{W}_{P_2}, \mathcal{W}_{Q_1}, \mathcal{W}_{Q_2}, \mathcal{W}_{R_1}, \mathcal{W}_{R_2}, \mathcal{W}_Z, \mathcal{W}_F, \mathcal{W}_D, \mathcal{W}_{D_1}, \mathcal{W}_{D_2}, \mathcal{W}_{FD}, \mathcal{W}_{FD_1}$, and \mathcal{W}_{FD_2} are defined in Theorem 2, and

$$\begin{aligned} \bar{\mathcal{P}} &\triangleq \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{P} \end{bmatrix}, \quad \bar{\mathcal{Q}}_1 \triangleq \begin{bmatrix} \mathcal{Q}_1 & 0 \\ 0 & -\mathcal{Q}_1 \end{bmatrix} \\ \bar{\mathcal{Q}}_2 &\triangleq \begin{bmatrix} \mathcal{Q}_2 & 0 \\ 0 & -\mathcal{Q}_2 \end{bmatrix}, \quad \bar{\mathcal{R}}_{11} = \bar{\mathcal{R}}_{21} \triangleq \begin{bmatrix} \tau \mathcal{R}_1 & 0 \\ 0 & -\frac{1}{\tau} \mathcal{R}_1 \end{bmatrix} \\ \bar{\mathcal{R}}_{12} &\triangleq \begin{bmatrix} (d_2 - m\tau) \mathcal{R}_2 & 0 & 0 \\ 0 & -\frac{2}{d_2 - m\tau} \mathcal{R}_2 & 0 \\ 0 & 0 & -\frac{1}{d_2 - m\tau} \mathcal{R}_2 \end{bmatrix} \\ \bar{\mathcal{R}}_{22} &\triangleq \begin{bmatrix} (d_2 - m\tau) \mathcal{R}_2 & 0 & 0 \\ 0 & -\frac{1}{d_2 - m\tau} \mathcal{R}_2 & 0 \\ 0 & 0 & -\frac{2}{d_2 - m\tau} \mathcal{R}_2 \end{bmatrix} \\ \bar{\mathcal{Z}} &\triangleq \begin{bmatrix} (d_2 - m\tau + 1) \mathcal{Z} & 0 \\ 0 & -\mathcal{Z} \end{bmatrix} \\ \mathcal{W}_M &\triangleq \begin{bmatrix} \mathcal{M} & \mathcal{S} & 0_{n, (m+3)n} & \mathcal{N} \end{bmatrix} \\ \mathcal{W}_S &\triangleq \begin{bmatrix} C - I_n & 0_{n, (m+2)n} & A & A_d - I_n \end{bmatrix}. \end{aligned}$$

IV. SYNCHRONIZATION

In the previous section, we have analyzed the stability for a single neural network. In this section, we will consider the synchronization problems.

A. Simple Case

Let us consider the network (2) as a drive system for synchronization problem, then, we construct the response system as follows:

$$\begin{aligned} y(k+1) = & C(\alpha_k) y(k) + A(\alpha_k) g(y(k)) \\ & + A_d(\alpha_k) g(y(k-d(k))) + J + u(k) \end{aligned} \quad (42)$$

for $k = 1, 2, \dots$, where $y(k) \triangleq [y_1(k), y_2(k), \dots, y_n(k)]^T \in \mathbb{R}^n$ is the neuron state vector of the response system and $u(k) \in \mathbb{R}^n$ is feedback control input for synchronization to be designed later.

Define $e(k) \triangleq x(k) - y(k)$, $h(e(k)) \triangleq g(x(k)) - g(y(k))$, and $h(e(k-d(k))) \triangleq g(x(k-d(k))) - g(y(k-d(k)))$. By Assumptions 1 and 2, it is easily verified from (3) that $h_i(e_i(k))$ satisfies $h_i(0) = 0$ and $\forall e_i \neq 0$

$$\lambda_i^- \leq \frac{h_i(e_i(k))}{e_i(k)} \leq \lambda_i^+ \quad \forall i = 1, 2, \dots, n. \quad (43)$$

Hence, the error dynamic system is written by

$$\begin{aligned} e(k+1) = & C(\alpha_k) e(k) + A(\alpha_k) h(e(k)) \\ & + A_d(\alpha_k) h(e(k-d(k))) - u(k). \end{aligned} \quad (44)$$

The control input associated with the state feedback is designed as

$$u(k) = K(\alpha_k) e(k) \quad (45)$$

where $K(\alpha_k) \in \mathbb{R}^{n \times n}$ is the gain matrix to be determined for synchronizing both a drive system and response system. Here, the parameter matrices $K(\alpha_k)$ are switching with the same switching signal as the original system. Then, substituting $u(k)$ in (45) into the error dynamic system (44), we obtain the closed-loop system as follows:

$$\begin{aligned} e(k+1) = & (C(\alpha_k) - K(\alpha_k)) e(k) \\ & + A(\alpha_k) h(e(k)) + A_d(\alpha_k) h(e(k-d(k))). \end{aligned} \quad (46)$$

With Theorem 2, we have the following controller design result.

Theorem 3: Given an integer $m \geq 1$, $\tau \geq 1$, $m\tau < d_2$, and a constant $0 < \beta < 1$, supposed that there exist scalars σ_j , $j = 1, 2, 3$, positive definite matrices $\mathcal{P}(i) \in \mathbb{R}^{n \times n}$, $\mathcal{Q}_1(i) \in \mathbb{R}^{mn \times mn}$, $\mathcal{Q}_2(i) \in \mathbb{R}^{n \times n}$, $\mathcal{Q}_3(i) \in \mathbb{R}^{n \times n}$, $\mathcal{R}_1(i) \in \mathbb{R}^{n \times n}$, $\mathcal{R}_2(i) \in \mathbb{R}^{n \times n}$, $\mathcal{Z}(i) \in \mathbb{R}^{n \times n}$, $\Pi = \text{diag}\{\pi_1, \pi_2, \dots, \pi_n\}$, $V = \text{diag}\{v_1, v_2, \dots, v_n\}$, and matrices $H(i)$, $L(i)$ such that for $i \in \mathcal{N}$

$$\begin{aligned} \bar{\Phi}_1(i) \triangleq & \mathcal{W}_P^T \bar{\mathcal{P}}(i) \mathcal{W}_P + \sum_{i=1}^2 \mathcal{W}_{Q_i}^T \bar{\mathcal{Q}}_i(i) \mathcal{W}_{Q_i} \\ & + \sum_{i=1}^2 \mathcal{W}_{R_i}^T \bar{\mathcal{R}}_{1i}(i) \mathcal{W}_{R_i} + \mathcal{W}_Z^T \bar{\mathcal{Z}}(i) \mathcal{W}_Z \\ & + \text{sym} \left\{ \mathcal{W}_{P_1}^T \mathcal{P}(i) \mathcal{W}_{P_2} + \mathcal{W}_\sigma^T \mathcal{W}_H \right. \\ & + \mathcal{W}_F^T \Pi (\Lambda^+ + \Lambda^-) \mathcal{W}_{P_1} - \mathcal{W}_{P_1}^T \Pi \Lambda^+ \Lambda^- \mathcal{W}_{P_1} \\ & - \mathcal{W}_F^T \Pi \mathcal{W}_F + \mathcal{W}_{FD}^T V (\Lambda^+ + \Lambda^-) \mathcal{W}_{FD} \\ & \left. - \mathcal{W}_D^T V \Lambda^+ \Lambda^- \mathcal{W}_D - \mathcal{W}_{FD}^T V \mathcal{W}_{FD} \right\} < 0 \end{aligned} \quad (47)$$

$$\begin{aligned} \bar{\Phi}_2(i) \triangleq & \mathcal{W}_P^T \bar{\mathcal{P}}(i) \mathcal{W}_P + \sum_{i=1}^2 \mathcal{W}_{Q_i}^T \bar{\mathcal{Q}}_i(i) \mathcal{W}_{Q_i} \\ & + \sum_{i=1}^2 \mathcal{W}_{R_i}^T \bar{\mathcal{R}}_{2i}(i) \mathcal{W}_{R_i} + \mathcal{W}_Z^T \bar{\mathcal{Z}}(i) \mathcal{W}_Z \\ & + \text{sym} \left\{ \mathcal{W}_{P_1}^T \mathcal{P}(i) \mathcal{W}_{P_2} + \mathcal{W}_\sigma^T \mathcal{W}_H \right. \\ & + \mathcal{W}_F^T \Pi (\Lambda^+ + \Lambda^-) \mathcal{W}_{P_1} - \mathcal{W}_{P_1}^T \Pi \Lambda^+ \Lambda^- \mathcal{W}_{P_1} \\ & - \mathcal{W}_F^T \Pi \mathcal{W}_F + \mathcal{W}_{FD}^T V (\Lambda^+ + \Lambda^-) \mathcal{W}_{FD} \\ & \left. - \mathcal{W}_D^T V \Lambda^+ \Lambda^- \mathcal{W}_D - \mathcal{W}_{FD}^T V \mathcal{W}_{FD} \right\} < 0 \end{aligned} \quad (48)$$

where

$$\mathcal{W}_\sigma \triangleq [\sigma_1 I_n \quad \sigma_2 I_n \quad 0_{n,(m+3)n} \quad \sigma_3 I_n]$$

$$\mathcal{W}_{P1} \triangleq [I_n \quad 0_{n,(m+5)n}]$$

$$\mathcal{W}_S(i) \triangleq [H(i)(C(i) - I_n) - L(i) \quad 0_{n,(m+2)n} \quad H(i)A(i) \\ H(i)A_d(i) - H(i)]$$

and other matrix notations are defined in Theorem 2. Then, the discrete-time switched DNN in (46) with time-delay is exponentially stable for any switching signal with average dwell time satisfying $T_a > T_a^* = \text{ceil}(-\ln \mu / \ln \beta)$, where $\mu \geq 1$ satisfies (29). Moreover, the controller gain can be given by

$$K(i) = L(i)H^{-1}(i).$$

Proof: Substituting $C(i)$ in (27) with $C(i) - K(i)$, that is $\mathcal{W}_S(i)$ in (27) is replaced by

$$[C(i) - I_n - K(i) \quad 0_{n,(m+2)n} \quad A(i) \quad A_d(i) - I_n].$$

Then, setting $\mathcal{M}^T(i) = \sigma_1 H(i)$, $\mathcal{N}^T(i) = \sigma_2 H(i)$, and $\mathcal{S}^T(i) = \sigma_3 H(i)$ with $H(i)$ nonsingular in (27) and $L(i) = H(i)K(i)$, we have $\bar{\Phi}_1(i) < 0$ and $\bar{\Phi}_2(i) < 0$ from $\Phi_1(i) < 0$, $\Phi_2(i) < 0$, and $K(i) = L(i)H^{-1}(i)$. The proof is completed. ■

B. Coupled DNNs Case

In the previous parts, the exponential stability of a single neural network is studied. From now on, it will be the main focus to analyze the exponential synchronization problem for an array of coupled identical switched neural networks with time-delay. Consider a coupled system of q identical neural networks described by

$$\begin{aligned} z_j(k+1) &= C(\alpha_k)z_j(k) + A(\alpha_k)g(z_j(k)) \\ &\quad + A_d(\alpha_k)g(z_j(k-d(k))) \\ &\quad + D(\alpha_k) \sum_{l=1}^q \omega_{jl} z_l(k), \quad j = 1, 2, \dots, q \end{aligned} \quad (49)$$

where $z_j(k) = [z_{j1}, z_{j2}, \dots, z_{jn}]^T$ is the state vector of the j th neural network, $D(\alpha_k)$ is the linking matrix, and $W = [\omega_{jl}]_{q \times q}$ is the coupled configuration matrix of the network with $\omega_{jl} = \omega_{lj} \geq 0$ but not all zero. The diagonal element ω_{jj} is defined as follows:

$$\omega_{jj} = - \sum_{l=1, l \neq j}^q \omega_{jl} = - \sum_{l=1, l \neq j}^q \omega_{lj}, \quad j, l = 1, 2, \dots, q. \quad (50)$$

The coupling satisfying $\sum_{l=1}^q \omega_{jl} = 0$ is called as linear coupling.

For presentation convenience, we denote the following:

$$z(k) \triangleq [z_1^T(k) \quad z_2^T(k) \quad \dots \quad z_q^T(k)]^T$$

$$G(z(k)) \triangleq [g^T(z_1(k)) \quad g^T(z_2(k)) \quad \dots \quad g^T(z_q(k))]^T.$$

Using the Kronecker product, we can rewrite system (49) into a more compact form as follows:

$$\begin{aligned} z(k+1) &= (I \otimes C(\alpha_k) + W \otimes D(\alpha_k))z(k) \\ &\quad + (I \otimes A(\alpha_k))G(z(k)) \\ &\quad + (I \otimes A_d(\alpha_k))G(z(k-d(k))). \end{aligned} \quad (51)$$

Lemma 1: [29] Let $e \in \mathbb{R}^q$ with all components being 1 and $U = qI - ee^T = [u_{ij}]_{q \times q}$. For $P \in \mathbb{R}^{n \times n}$, $x = [x_1^T, x_2^T, \dots, x_q^T]^T$, and $y = [y_1^T, y_2^T, \dots, y_q^T]^T$ with $x_i, y_i \in \mathbb{R}^n$. Then, the following equations hold:

$$UW = WU = qW$$

$$x^T(U \otimes P)y = \sum_{1 \leq i < j \leq q} (x_i - x_j)^T P(y_i - y_j).$$

Theorem 4: Given integers $m \geq 1$, $0 < d_1 \leq d_2$ and a constant $0 < \beta < 1$, supposed that there exist positive definite matrices $\mathcal{P}(i) \in \mathbb{R}^{n \times n}$, $\mathcal{Q}_1(i) \in \mathbb{R}^{mn \times mn}$, $\mathcal{Q}_2(i) \in \mathbb{R}^{n \times n}$, $\mathcal{Q}_3(i) \in \mathbb{R}^{n \times n}$, $\mathcal{R}_1(i) \in \mathbb{R}^{n \times n}$, $\mathcal{R}_2(i) \in \mathbb{R}^{n \times n}$, $\mathcal{Z}(i) \in \mathbb{R}^{n \times n}$, $\Pi = \text{diag}\{\pi_1, \pi_2, \dots, \pi_n\}$, $V = \text{diag}\{v_1, v_2, \dots, v_n\}$, and matrices $\mathcal{M}(i)$, $\mathcal{N}(i)$, $\mathcal{S}(i)$ such that for $i \in \mathcal{N}$

$$\bar{\Phi}_{1jl}(i) < 0, \quad \bar{\Phi}_{2jl}(i) < 0, \quad 1 \leq j < l \leq q \quad (52)$$

where $\bar{\Phi}_{vjl}(i)$, $v = 1, 2$ are expressed by $\Phi_v(i)$, $v = 1, 2$ in (27) and (28) with $W_S(i)$ replaced by $W_{Sjl}(i) = [C(i) + q\omega_{jl}D(i) - I_n \quad 0_{n,(m+2)n} \quad A(i) \quad A_d(i) - I_n]$, respectively. Then, the coupled system in (51) with time-delay satisfying (A2) is exponentially synchronized for any switching signal with average dwell time satisfying $T_a > T_a^* = \text{ceil}(-\ln \mu / \ln \beta)$, where $\mu \geq 1$ satisfies

$$\begin{aligned} \mathcal{P}(i) &\leq \mu \mathcal{P}(j), \quad \mathcal{Q}_\varrho(i) \leq \mu \mathcal{Q}_\varrho(j), \quad \varrho = 1, 2, 3 \\ \mathcal{R}_v(i) &\leq \mu \mathcal{R}_v(j), \quad v = 1, 2 \quad \forall i, j \in \mathcal{N}. \end{aligned} \quad (53)$$

Proof: Consider the Lyapunov function as follows:

$$\bar{W}(z_k, \alpha_k) = \sum_{j=1}^4 \bar{W}_j(z_k, \alpha_k) \quad (54)$$

with

$$\begin{cases} \bar{W}_1(z_k, \alpha_k) \triangleq z^T(k)(U \otimes \mathcal{P})(\alpha_k)z(k) \\ \bar{W}_2(z_k, \alpha_k) \triangleq \sum_{l=k-\tau}^{k-1} \beta^{k-l} \varphi^T(l)(U \otimes \mathcal{Q}_1(\alpha_k))\varphi(l) \\ \quad + \sum_{l=k-d_2}^{k-1} \beta^{k-l} z^T(l)(U \otimes \mathcal{Q}_2(\alpha_k))z(l) \\ \bar{W}_3(z_k, \alpha_k) \triangleq \sum_{s=-\tau}^{-1} \sum_{l=k+s}^{k-1} \beta^{k-l} \varsigma^T(l)(U \otimes \mathcal{R}_1(\alpha_k))\varsigma(l) \\ \quad + \sum_{s=-d_2}^{-m\tau-1} \sum_{l=k+s}^{k-1} \beta^{k-l} \varsigma^T(l)(U \otimes \mathcal{R}_2(\alpha_k))\varsigma(l) \\ \bar{W}_4(z_k, \alpha_k) \triangleq \sum_{s=-d_2+1}^{-m\tau+1} \sum_{l=k-1+s}^{k-1} \beta^{k-l} z^T(l)(U \otimes \mathcal{Z}(\alpha_k))z(l) \end{cases}$$

where

$$\varphi(l) \triangleq \begin{bmatrix} z(l) \\ z(l-\tau) \\ z(l-2\tau) \\ \vdots \\ z(l-(m-1)\tau) \end{bmatrix}, \quad \varsigma(l) \triangleq z(l+1) - z(l)$$

and $\mathcal{P}(a_k) > 0$, $\mathcal{S}(a_k) > 0$, $\mathcal{Q}_1(a_k) > 0$, $\mathcal{Q}_2(a_k) > 0$, $\mathcal{R}_1(a_k) > 0$, and $\mathcal{R}_2(a_k) > 0$ are real matrices to be determined.

For $k \in [k_l, k_{l+1})$, as in the previous section, we define $\Delta \bar{W}_j(z_k, a_k) \triangleq \bar{W}_j(z_{k+1}, a_k) - \bar{W}_j(z_k, a_k)$, $j = 1, 2, 3, 4$. Similar to the proof in Section III-B, we have the following:

$$\Delta \bar{W}_1(z_k, a_k) = \varsigma^T(k)(U \otimes \mathcal{P}(a_k))\varsigma(k) + 2z^T(k)(U \otimes \mathcal{P}(a_k))\varsigma(k) \quad (55)$$

$$\begin{aligned} \Delta \bar{W}_2(z_k, a_k) &= -(1 - \beta) \sum_{l=k-\tau}^{k-1} \beta^{k-l} \varphi^T(l) \\ &\times (U \otimes \mathcal{Q}_1(a_k))\varphi(l) \\ &- (1 - \beta) \sum_{l=k-d_2}^{k-1} \beta^{k-l} z^T(l)(U \otimes \mathcal{Q}_2(a_k))z(l) \\ &+ \beta \varphi^T(k)(U \otimes \mathcal{Q}_1(a_k))\varphi(k) \\ &- \beta^{\tau+1} \varphi^T(k - \tau)(U \otimes \mathcal{Q}_1(a_k))\varphi(k - \tau) \\ &+ \beta z^T(k)(U \otimes \mathcal{Q}_2(a_k))z(k) \\ &- \beta^{d_2+1} z^T(k - d_2)(U \otimes \mathcal{Q}_2(a_k))z(k - d_2) \end{aligned} \quad (56)$$

$$\begin{aligned} \Delta \bar{W}_3(z_k, a_k) &\leq -(1 - \beta) \sum_{s=-\tau}^{-1} \sum_{l=k+s}^{k-1} \beta^{k-l} \\ &\times \varsigma^T(l)(U \otimes \mathcal{R}_1(a_k))\varsigma(l) \\ &- (1 - \beta) \sum_{s=-d_2}^{-m\tau-1} \sum_{l=k+s}^{k-1} \beta^{k-l} \varsigma^T(l)(U \otimes \mathcal{R}_2(a_k))\varsigma(l) \\ &- \frac{\beta^{1+\tau}}{\tau} [z(k) - z(k - \tau)]^T (U \otimes \mathcal{R}_1(a_k)) \\ &\times [z(k) - z(k - \tau)] - (1 + \gamma) \tilde{\beta} \varsigma_1^T(k)(U \otimes \mathcal{R}_2(a_k))\varsigma_1(k) \\ &- (2 - \gamma) \tilde{\beta} \varsigma_2^T(k)(U \otimes \mathcal{R}_2(a_k))\varsigma_2(k) \\ &+ \beta \tau \varsigma^T(k)(U \otimes \mathcal{R}_1(a_k))\varsigma(k) \\ &+ \beta(d_2 - m\tau) \varsigma^T(k)(U \otimes \mathcal{R}_2(a_k))\varsigma(k) \end{aligned} \quad (57)$$

$$\begin{aligned} \Delta \bar{W}_4(z_k, a_k) &\leq -(1 - \beta) \sum_{s=-d_2+1}^{-m\tau+1} \sum_{l=k-1+s}^{k-1} \beta^{k-l} \\ &\times z^T(l)(U \otimes \mathcal{Z}(a_k))z(l) \\ &+ \beta(d_2 - m\tau + 1) z^T(k)(U \otimes \mathcal{Z}(a_k))z(k) \\ &- \beta^{d_2+1} z^T(k - d(k))(U \otimes \mathcal{Z}(a_k))z(k - d(k)) \end{aligned} \quad (58)$$

where $\varsigma_1(k) \triangleq z(k - d(k)) - z(k - d_2)$ and $\varsigma_2(k) \triangleq z(k - m\tau) - z(k - d(k))$.

On the other hand, using Lemma 1, for any appropriately dimensioned matrices $\mathcal{M}(a_k)$, $\mathcal{N}(a_k)$, and $\mathcal{S}(a_k)$, $a_k \in \mathcal{N}$, the following equation is true:

$$\begin{aligned} 0 &= 2 \left[z^T(k)(U \otimes \mathcal{M}^T(a_k)) + \varsigma^T(k)(U \otimes \mathcal{N}^T(a_k)) \right. \\ &\quad \left. + z^T(k - \tau)(U \otimes \mathcal{S}^T(a_k)) \right] \\ &\times \{ [I \otimes C(a_k) + W \otimes D(a_k) - I] z(k) - \varsigma(k) \\ &\quad + (I \otimes A(a_k))G(z(k)) \\ &\quad + (I \otimes A_d(a_k))G(z(k - d(k))) \}. \end{aligned} \quad (59)$$

From Assumption 2, for any scalar $\pi_i > 0$ and $v_i > 0$, we have that for $i = 1, 2, \dots, n$ and $1 \leq j < l \leq q$

$$\left. \begin{aligned} &2(g(z_j(k)) - g(z_l(k)))^T \Pi \\ &\times (\Lambda^+ + \Lambda^-)(z_j(k) - z_l(k)) - 2(z_j(k) - z_l(k))^T \Pi \\ &\times \Lambda^+ \Lambda^- (z_j(k) - z_l(k)) \\ &- 2(g(z_j(k)) - g(z_l(k)))^T \Pi (g(z_j(k)) - g(z_l(k))) \geq 0 \\ &2(g(z_j(k - d(k))) - g(z_l(k - d(k))))^T \\ &\times V(\Lambda^+ + \Lambda^-)(z_j(k - d(k)) - z_l(k - d(k))) \\ &- 2(z_j(k - d(k)) - z_l(k - d(k)))^T \\ &\times V \Lambda^+ \Lambda^- (z_j(k - d(k)) - z_l(k - d(k))) \\ &- 2(g(z_j(k - d(k))) - g(z_l(k - d(k))))^T \\ &\times V(g(z_j(k - d(k))) - g(z_l(k - d(k)))) \geq 0 \end{aligned} \right\} \quad (60)$$

where Π , V , Λ^+ , and Λ^- are defined in (38). It follows from (55)–(60) and Lemma 1 that the following inequality holds:

$$\begin{aligned} &\Delta \bar{W}(z_k, a_k) + (1 - \beta) \bar{W}(z_k, a_k) \\ &\leq \sum_{1 \leq j < l \leq q} \tilde{\phi}_{jl}^T(k) [\gamma \tilde{\Phi}_{1jl}(a_k) + (1 - \gamma) \tilde{\Phi}_{2jl}(a_k)] \tilde{\phi}_{jl}(k) \end{aligned}$$

where

$$\begin{aligned} \varsigma_{1j}(k) &\triangleq z_j(k - d(k)) - z_j(k - d_2) \\ \varsigma_{2j}(k) &\triangleq z_j(k - m\tau) - z_j(k - d(k)) \\ \tilde{\phi}_{jl}(k) &\triangleq [(\varphi_j(k) - \varphi_l(k))^T (z_j(k - d_1) - z_l(k - d_1))^T \\ &\quad (z_j(k - d(k)) - z_l(k - d(k)))^T \\ &\quad (z_j(k - d_2) - z_l(k - d_2))^T \\ &\quad (G(z_j(k)) - G(z_l(k)))^T \\ &\quad (G(z_j(k - d(k))) - G(z_l(k - d(k))))^T \\ &\quad (\varsigma_j(k) - \varsigma_l(k))]^T \\ \varphi_j(k) &\triangleq [z_j^T(k) z_j^T(k - \tau) \dots z_j^T(k - (m - 1)\tau)]^T \\ \varsigma_j(k) &\triangleq z_j(k + 1) - z_j(k) \end{aligned}$$

and $\tilde{\Phi}_{1jl}(a_k)$ and $\tilde{\Phi}_{2jl}(a_k)$ are defined in (53). Moreover, from $\tilde{\Phi}_{1jl}(a_k) < 0$ and $\tilde{\Phi}_{2jl}(a_k) < 0$, it follows that $\forall k \in [k_l, k_{l+1})$:

$$\begin{aligned} &\Delta \bar{W}(z_k, a_k) + (1 - \beta) \bar{W}(z_k, a_k) \\ &\leq \tilde{\phi}_{jl}^T(k) [\gamma \tilde{\Phi}_{1jl}(a_k) + (1 - \gamma) \tilde{\Phi}_{2jl}(a_k)] \tilde{\phi}_{jl}(k) < 0. \end{aligned}$$

The rest of the proof can be followed by the same lines of the proof of Theorem 1. The proof is completed. ■

V. ILLUSTRATIVE EXAMPLES

In this section, we shall present three examples to show the effectiveness of the methods proposed in the previous sections.

Example 1: Consider the discrete-time switched DNN in (5) with $N = 2$ and its parameters given as follows:

$$\begin{aligned} C(1) &= \text{diag}\{0.01, 0.3\}, \quad C(2) = \text{diag}\{0.01, 0.25\} \\ A(1) &= \begin{bmatrix} -0.1 & 0 \\ 0.1 & 0.005 \end{bmatrix}, \quad A_d(1) = \begin{bmatrix} -0.1 & 0.2 \\ -0.2 & -0.1 \end{bmatrix} \\ A(2) &= \begin{bmatrix} -0.15 & 0 \\ 0.1 & -0.015 \end{bmatrix}, \quad A_d(2) = \begin{bmatrix} 0.1 & -0.1 \\ -0.3 & -0.1 \end{bmatrix} \end{aligned}$$

TABLE I
MAXIMUM ALLOWABLE DELAYS FOR DIFFERENT m

Methods	d_{\max}
Theorem 1, $m = 1, \tau = 9$	9
Theorem 1, $m = 3, \tau = 3$	9
Theorem 1, $m = 5, \tau = 2$	10
Theorem 1, $m = 10, \tau = 1$	10

and $\Lambda^+ = \text{diag}\{1, 1\}$, $\Lambda^- = \text{diag}\{0, 0\}$. Initially, we consider the constant time-delay case. Our attention is focused on finding the maximum allowable delay d_{\max} such that the discrete-time switched DNN in (5) is exponentially stable. We choose $\beta = 0.8$, and the results derived by Theorem 1 for different m are listed in Table I. As shown in Table I, the maximum allowable delay becomes larger as the partitioning becomes thinner.

Next, we consider the average dwell time scheme. Set $\mu = 1.5 > 1$, thus $T_a > T_a^* = \text{ceil}(-\ln \mu / \ln \beta) = 2$. Solving LMIs (7)–(8) in Theorem 1 with $m = 5$, we can obtain a set of solutions for $P(1)$, $P(2)$, $Q(1)$, $Q(2)$, $R(1)$, and $R(2)$, which means that the above discrete-time switched DNN is exponentially stable. Taking $T_a = 3 > T_a^*$, and considering (10) yield $a = 749.9$, $b = 5052.9$, $\eta = 2.5958$, and $\rho = 0.975$, thus

$$\|\zeta(k)\| \leq 2.5958 \times 0.975^{(k-k_0)} \|\zeta(k_0)\|_{C^1}.$$

Now, we consider that $\mu > 1$ and $\beta \rightarrow 1$, obviously, we have $T_a \rightarrow \infty$, that is, there is no switching. In such a case, the discrete-time switched DNN in (5) is effectively operating at one of the subsystems all the time, and it turns out to be the form of (25). The related result for (25) is shown in Corollary 1. To check Corollary 1 and compare it with some existing results, we give the following parameters:

$$C = \begin{bmatrix} 0.7 & 0 \\ 0 & -0.7 \end{bmatrix}, \quad A = \begin{bmatrix} 0.01 & 0.22 \\ -0.02 & 0.05 \end{bmatrix},$$

$$A_d = \begin{bmatrix} 0.2 & 0.3 \\ -0.2 & -0.1 \end{bmatrix}$$

and $\Lambda^+ = \text{diag}\{1, 1\}$, $\Lambda^- = \text{diag}\{0, 0\}$.

The comparison results are given in Table II, from which we can see that when $m = 1$ (that is, no delay partitioning) the allowable maximum delay d are all $d = 23$ by the methods in [39], [40], [55] and Corollary 1 of this paper. The number of the variables to be determined in our result (Corollary 1) is, however, fewer than that in [39], [40], and [55], that is, some variables in [39], [40], and [55] are redundant. Thus, from the viewpoint of computational complexity, Corollary 1 in this paper is more powerful. In addition, we can also see from Table II that when $m = 2$ and $m = 3$, the allowable maximum delay d are $d = 44$ and $d = 45$, respectively, by both methods in Corollary 1 and [39]. The number of the variables in Corollary 1 is, however, fewer than that of [39]; thus our result in Corollary 1 improves that in [39].

Example 2: In this example, we will consider the non-switching case (which is a special case, see Remark 4), that is, there is no switching in the considered discrete-time DNN.

The related results are given in Corollaries 1 and 3 for constant and time-varying delay cases, respectively.

Recently, there are some new results reported on the stability of discrete-time DNNs, see [39], [40], [46], [55], and so on. In the following, we will compare our result in Corollary 3 with some of them to demonstrate the reduced conservatism of the developed approach in this paper. Now, consider the non-switched discrete-time DNN in (39) with time-varying delay $d(k)$, and for convenience, we choose the same parameters as those in [39], that is

$$C = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \quad A = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix},$$

$$A_d = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}$$

and take activation functions $g_1(s) = \tanh(s)$, $g_2(s) = \tanh(s)$. It can be verified that Assumption 2 is satisfied with $\lambda_1^- = \lambda_2^- = 0$ and $\lambda_1^+ = \lambda_2^+ = 1$. The results derived by Corollary 3 for different m are listed in Table III. In addition, the related results using the methods in [39], [40], [46], and [55] are displayed in Table III for comparison. Table III shows that the new criterion given in Corollary 3 is less conservative than the previous results.

Example 3: In this example, we will check the result for the synchronization problem proposed in Theorem 3. Consider the following switched DNN with time-varying delay and $N = 2$:

$$C(1) = \text{diag}\{1, 0.3\}, \quad C(2) = \text{diag}\{-0.1, 0.4\}$$

$$A(1) = \begin{bmatrix} -0.1 & 0.3 \\ 0.5 & 0 \end{bmatrix}, \quad A_d(1) = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}$$

$$A(2) = \begin{bmatrix} 0.1 & -0.2 \\ 0 & -0.1 \end{bmatrix}, \quad A_d(2) = \begin{bmatrix} 0 & 0.2 \\ 0.2 & -0.1 \end{bmatrix}.$$

For given $\beta = 0.8$, $\mu = 4$, $d_1 = 4$, $d_2 = 5$, $\sigma_1 = 1$, $\sigma_2 = 0.2$, and $\sigma_3 = 3$, the corresponding feasible solutions to the LMIs (47) and (48) can be calculated as $T_a^* = \text{ceil}(-\ln \mu / \ln \beta) = 7$ and

$$H(1) = \begin{bmatrix} 187.5640 & -14.1506 \\ 20.9342 & 97.1493 \end{bmatrix}$$

$$L(1) = \begin{bmatrix} 122.8718 & 30.3305 \\ 32.0865 & 1.1535 \end{bmatrix}$$

$$H(2) = \begin{bmatrix} 166.8438 & -5.5504 \\ 38.4899 & 101.0527 \end{bmatrix}$$

$$L(2) = \begin{bmatrix} 98.3644 & 20.7151 \\ 36.1095 & 6.4222 \end{bmatrix}$$

$$K(1) = \begin{bmatrix} 0.6103 & 0.4011 \\ 0.1670 & 0.0362 \end{bmatrix}$$

$$K(2) = \begin{bmatrix} 0.5355 & 0.2344 \\ 0.1992 & 0.0745 \end{bmatrix}.$$

We choose $T_a = 10 > T_a^*$. The switched signal is shown in Fig. 1 (here, 1 and 2 represent the first and second subsystems, respectively), and Fig. 2 shows the synchronization error with the time-varying delay as follows:

$$d(k) = \begin{cases} 4, & 0 \leq k < 10 \\ 5, & 10 \leq k < 20 \\ 4, & \text{else} \end{cases}$$

TABLE II
ALLOWABLE MAXIMUM DELAY d OBTAINED BY DIFFERENT METHODS

Methods	[40]	[55]	$m = 1, \tau = 32$		$m = 2, \tau = 22$		$m = 5, \tau = 9$	
			[39]	Corollary 1	[39]	Corollary 1	[39]	Corollary 1
d	32	32	32	32	44	44	45	45
variables	44	29	68	25	98	32	236	77

TABLE III
MAXIMUM ALLOWABLE DELAYS

Methods	$d_1 = 4$	$d_1 = 6$	$d_1 = 8$	$d_1 = 10$	$d_1 = 15$
Theorem 2 [40]	11	12	13	14	17
Theorem 1 [55]	12	13	14	16	19
Theorem 1 [46]	16	17	18	19	22
Theorem 1 [39]	16	17	18	19	22
Corollary 3	16	17	19	21	26
	$m = 2, \tau = 2$	$m = 2, \tau = 3$	$m = 4, \tau = 2$	$m = 5, \tau = 2$	$m = 5, \tau = 3$

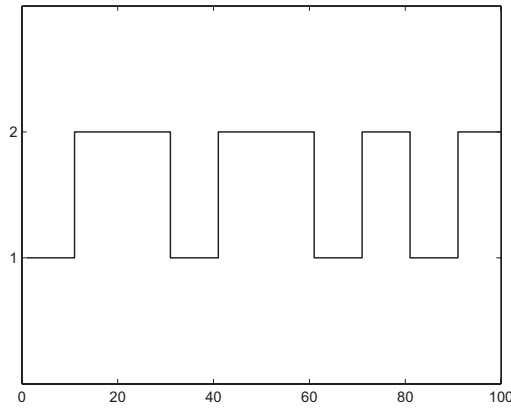


Fig. 1. Switching signal.

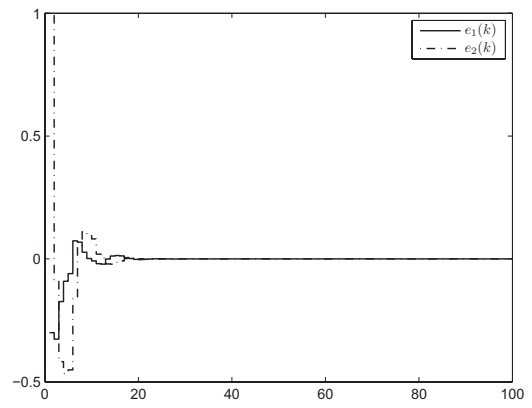


Fig. 2. Synchronization error.

and the initial conditions $e(-k) = [k \ 10/k]^T, k = 1, \dots, 5$, $e(0) = [1.0 \ -0.3]^T$.

Example 4: In this example, the effectiveness of the Theorem 4 for synchronization of the coupled neural network is illustrated. Consider the coupled neural network in (51) of multiple identical switched neural networks. For simplicity, we consider $N = 2$ and $q = 3$. The system parameters are given as follows:

$$\begin{aligned}
 C(1) &= \begin{bmatrix} 0.3 & 0 \\ 0 & 0.4 \end{bmatrix} \\
 C(2) &= \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix} \\
 A(1) &= \begin{bmatrix} 2 & -0.2 \\ 0.2 & -1 \end{bmatrix} \\
 A(2) &= \begin{bmatrix} 0.1 & 0 \\ 0 & 0.5 \end{bmatrix} \\
 A_d(1) &= \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix} \\
 A_d(2) &= \begin{bmatrix} 0.1 & -0.1 \\ 0.02 & 0.1 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 W &= \begin{bmatrix} -0.2 & 0.1 & 0.1 \\ 0.1 & -0.2 & 0.1 \\ 0.1 & 0.1 & -0.2 \end{bmatrix} \\
 D(1) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
 D(2) &= \begin{bmatrix} 1.2 & 0 \\ 0 & 0.8 \end{bmatrix}.
 \end{aligned}$$

For given $\beta = 0.8$, $\mu = 2.5$, $d_1 = 2$, $d_2 = 4$, $g_1(s) = \tanh(-0.6s)$, and $g_2(s) = \tanh(0.4s)$. It is easy to check that

$$\Lambda^- = \begin{bmatrix} -0.6 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Lambda^+ = \begin{bmatrix} 0 & 0 \\ 0 & 0.4 \end{bmatrix}.$$

With the above parameters, by solving the LMIs in (52), we can find a feasible set of solutions as follows:

$$\begin{aligned}
 \mathcal{P}(1) &= \begin{bmatrix} 33.2064 & -3.0118 \\ -3.0118 & 19.0231 \end{bmatrix} \\
 \mathcal{P}(2) &= \begin{bmatrix} 127.8592 & 6.5062 \\ 6.5062 & 76.2237 \end{bmatrix}
 \end{aligned}$$

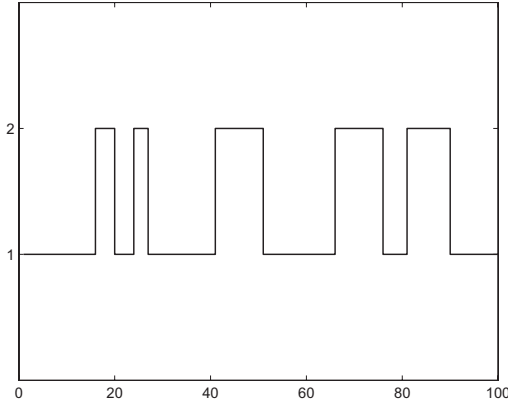


Fig. 3. Switching signal.

$$\begin{aligned}
 \mathcal{Q}_1(1) &= \begin{bmatrix} 1.0885 & -1.5329 & -0.3412 & 0.9409 \\ -1.5329 & 4.2380 & 0.8592 & -2.9695 \\ -0.3412 & 0.8592 & 0.6725 & -1.1648 \\ 0.9409 & -2.9695 & -1.1648 & 3.8186 \end{bmatrix} \\
 \mathcal{Q}_1(2) &= \begin{bmatrix} 21.3964 & 1.4782 & -21.6174 & -2.1276 \\ 1.4782 & 16.7526 & -0.8211 & -9.7534 \\ -21.6174 & -0.8211 & 23.2874 & 2.7222 \\ -2.1276 & -9.7534 & 2.7222 & 15.1141 \end{bmatrix} \\
 \mathcal{Q}_2(1) &= \begin{bmatrix} 0.5146 & -1.3995 \\ -1.3995 & 5.4649 \end{bmatrix} \\
 \mathcal{Q}_2(2) &= \begin{bmatrix} 19.1107 & -1.5184 \\ -1.5184 & 29.2242 \end{bmatrix} \\
 \mathcal{R}_1(1) &= \begin{bmatrix} 0.6520 & -1.8221 \\ -1.8221 & 6.3147 \end{bmatrix} \\
 \mathcal{R}_1(2) &= \begin{bmatrix} 50.9113 & 3.4783 \\ 3.4783 & 26.6173 \end{bmatrix} \\
 \mathcal{R}_2(1) &= \begin{bmatrix} 0.3762 & -1.1127 \\ -1.1127 & 5.0035 \end{bmatrix} \\
 \mathcal{R}_2(2) &= \begin{bmatrix} 43.3051 & 7.4848 \\ 7.4848 & 28.6740 \end{bmatrix} \\
 \mathcal{L}(1) &= \begin{bmatrix} 4.2222 & 0.2016 \\ 0.2016 & 0.1291 \end{bmatrix} \\
 \mathcal{L}(2) &= \begin{bmatrix} 1.8185 & 0.4048 \\ 0.4048 & 1.9783 \end{bmatrix} \\
 \Pi &= \begin{bmatrix} 78.8551 & 0 \\ 0 & 54.8507 \end{bmatrix} \\
 H &= \begin{bmatrix} 4.7727 & 0 \\ 0 & 11.6167 \end{bmatrix}.
 \end{aligned}$$

According to Theorem 4, the coupled discrete-time DNNs in (51) with the given parameters are synchronized. We choose $T_a = 10 > T_a^*$. The switched signal is shown in Fig. 3. The synchronization errors $e_{12}(k) = \|z_1(k) - z_2(k)\|$, $e_{13}(k) = \|z_1(k) - z_3(k)\|$, and $e_{23}(k) = \|z_2(k) - z_3(k)\|$ are given in Fig. 4, which shows that the synchronization error approaches zero.

VI. CONCLUSION

In this paper, we introduced a class of discrete-time DNNs with switching parameters as well as time-varying delay.

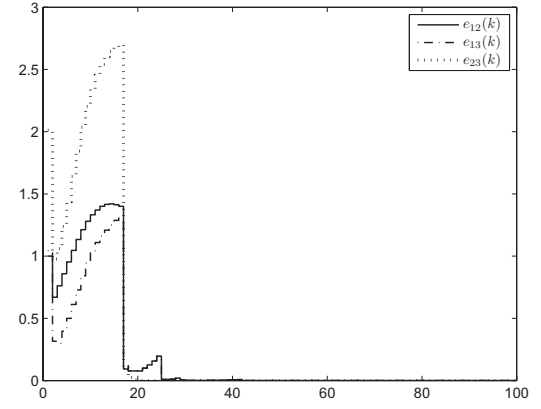


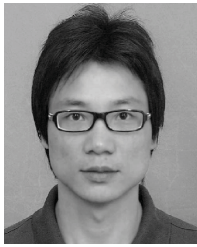
Fig. 4. Synchronization error.

We dealt with the stability analysis and the synchronization problems. The obtained stability results were based on the use of the average dwell time approach and the piecewise Lyapunov function technique. By considering the advantage of the delay-partitioning technique, a novel LKF, in combination with the free-weighting matrix technique, was introduced to arrive at the sufficient conditions that warranted the exponential stability of the switched neural networks with constant or time-varying delays. The obtained delay-dependent results also relied upon the partitioning so that the conservatism can be reduced. Then, we turned to the synchronization problem. It was shown that the addressed synchronization problem was solvable if several LMIs were feasible. Finally, the effectiveness of the proposed theory was illustrated by numerical examples.

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